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TRANSPORT SOLUTIONS TO THE ONE-DIMENSIONAL CRITICAL PROBLEM

by

George J. Mitsis

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ABSTRACT

The new method of Case(1) for treating the one-velocity transport equation is applied to a uniform, one-dimensional multiplying medium. The method leads to exact expressions for the neutron distribution and criticality conditions. These expressions depend on expansion coefficients which are shown to satisfy a Fredholm integral equation. The results of diffusion theory with the exact Milne problem extrapolation distance are shown to correspond to the zeroth-order approximation of the Neumann series solution to the Fredholm equation.

I. INTRODUCTION

Although the neutron transport equation has been extensively treated both analytically and numerically, exact solutions are available only for highly idealized cases. Moreover, the various computational techniques in common use are based on approximations of the exact equations or exact boundary conditions rather than its solutions.

A fresh approach to the problem was recently introduced by K. M. Case.(1) This new approach is based on the fact that, at least under certain restrictions, the transport equation is separable. Thus the familiar procedure of expanding the neutron distribution in the set of normal modes generated by the separation of variables and finding the expansion coefficients from the boundary conditions can be used.

The unique feature of this method is found in the singular nature of the expansion modes. The correct interpretations of such singularities were first given by Van Kampen(2) in his work on plasma oscillations.

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*A brief summary of the results of this paper as well as the results obtained independently by Dr. R. Zelazny at the Institute of Nuclear Research, Poland, were reported by Case.(4)

Van Kampen notes that it is permissible to use the principal value of integrals involving such eigenfunctions provided the contribution of the singular point is not neglected. These contributions can be accounted for through the addition of arbitrary multiples of the Dirac δ-function. The resulting expansion modes will therefore be highly singular, but integrable. Their singular form is of no concern since the final results are always expressed in terms of integrals. Furthermore, Case has shown that the appropriate singular modes for the one-speed transport equation form a complete set. Consequently, the neutron distribution is completely represented by an expansion in terms of these functions.

The purpose of this paper is to apply this method to the solution of the transport equation in a one-dimensional, uniform, multiplying medium. The treatment will be further restricted to a single energy and isotropic scattering. Under these assumptions, the angular neutron density ψ depends on the single position variable x and direction variable $\Omega_x = \mu$, and satifies the equation

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \Sigma_t \psi(x,\mu) = \frac{1}{2} \frac{\Sigma_s + \nu\Sigma_f}{\Sigma_t} \int_{-1}^{1} \psi(x,\mu')d\mu' \quad , \tag{1}$$

where the Σ's denote the macroscopic cross sections.

Introducing distance in mean free paths,

$$x' = \Sigma_t x \quad ,$$

the mean number of secondaries per collision

$$c = (\Sigma_s + \nu\Sigma_f)/\Sigma_t \quad ,$$

and dropping primes gives

$$\mu \frac{\partial \psi}{\partial x} + \psi = \frac{c}{2} \int_{-1}^{1} \psi(x,\mu')\, d\mu' \quad . \tag{2}$$

The general solution of Eq. (2) will be found by expanding $\psi(x,\mu)$ in the appropriate set of singular eigenfunctions. The results will be in the form of expressions for $\psi(x,\mu)$, the neutron density

$$\rho(x) = \int_{-1}^{1} \psi(x,\mu)\, d\mu \quad , \tag{3}$$

The neutron current

$$j(x) = \int_{-1}^{1} \mu \psi (x,\mu)\, d\mu \quad , \tag{3a}$$

and the critical condition for the system. All these expressions will be exact in the sense that they will depend on the expansion coefficients, which obey a Fredholm integral equation of the second kind. Successive approximations of the integral equation can be readily obtained and lead to recognizable forms of the respective expressions. In particular, the zeroth approximation of the coefficients will be shown to correspond to the results of diffusion theory.

II. EIGENFUNCTIONS OF THE TRANSPORT EQUATION

Inasmuch as the development of the eigenfunctions of Eq. (2) has been discussed in detail by Case,[1] it is sufficient to summarize the pertinent results here.

Separation of the variables x and μ shows that the solutions of Eq. (2) have the form

$$\psi(x,\mu) = e^{-x/\nu} \phi_\nu (\mu) \quad , \tag{4}$$

where $\phi_\nu (\mu)$ satisfies the equation

$$[1 - (\mu/\nu)]\, \phi_\nu(\mu) = \frac{c}{2} \int_{-1}^{1} \phi_\nu(\mu')\, d\mu' \quad . \tag{5}$$

If we use the normalization

$$\int_{-1}^{1} \phi_\nu (\mu)\, d\mu = 1 \quad , \tag{6}$$

Eq. (5) reduces to

$$(\nu - \mu)\, \phi_\nu(\mu) = \left(\frac{c}{2}\right) \nu \quad . \tag{7}$$

Since ν is not restricted, it can take values in the interval [-1, 1] where μ is defined, thus introducing singularities at the points $\mu = \nu$. The general solution of (7) must be written as

$$\phi_\nu (\mu) = \frac{c}{2}\, P \frac{\nu}{\nu - \mu} + \lambda(\nu)\, \delta(\mu - \nu) \quad , \tag{8}$$

where P denotes principal values of integrals involving $\phi_\nu(\mu)$ and the second term contributes only at the singular points $\mu = \nu$. Furthermore, Eq. (8) is a solution of (7) for any $\lambda(\nu)$.

For values of ν not in the interval $[-1,1]$, Eq. (8) reduces to

$$\phi_\nu(\mu) = \frac{c}{2}\,\frac{\nu}{\nu-\mu} \quad , \quad \nu \notin [1,-1] \quad ,$$

and the normalization condition (6) determines the values of ν:

$$1 = c\,\nu \tanh^{-1}\left(\frac{1}{\nu}\right) \quad . \tag{9}$$

There are two roots of Eq. (9) which are purely imaginary for $c > 1$. These roots will be denoted by $\pm\nu_0$ and the corresponding eigenfunctions by

$$\phi_{0\pm} = \frac{c}{2}\,\frac{\nu_0}{\nu_0 \mp \mu} \quad . \tag{10}$$

For ν in the interval $[-1,1]$, the normalization serves only to determine the form of $\lambda(\nu)$, leaving ν unrestricted. Thus,

$$\lambda(\nu) = 1 - c\nu \tanh^{-1} \nu \quad . \tag{11}$$

The eigenfunctions of Eq. (2) can now be separated into two classes:

a) For $\nu \notin [-1,1]$ there are two discrete eigenfunctions:

$$\psi_{0\pm}(x,\mu) = \phi_{0\pm}(\mu)\, e^{\mp x/\nu_0} \quad , \tag{12a}$$

with ν_0 and $\phi_{0\pm}(\mu)$ defined by equations (9) and (10), respectively.

b) For $\nu \in [-1,1]$, there is a continuum of solutions:

$$\psi_\nu(x,\mu) = \phi_\nu(\mu)\, e^{-x/\nu} \quad , \tag{12b}$$

where $\phi_\nu(\mu)$ is given by Eq. (8).

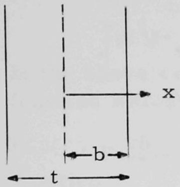
Finally, the general solution of Eq. (2) can be written as

$$\psi(x,\mu) = a_{0+}\psi_{0+}(x,\mu) + a_{0-}\psi_{0-}(x,\mu) + \int_{-1}^{1} A(\nu)\,\psi_\nu(x,\mu)\,d\nu \quad , \tag{13}$$

where $a_{0\pm}$ and $A(\nu)$ are arbitrary expansion coefficients.

III. APPLICATION TO THE CRITICAL PROBLEM

We now apply the above solution to a uniform, multiplying system, finite only in the x-direction. The thickness t and half-thickness b (see accompanying diagram) are in units of mean free paths and the origin of coordinates is at the center.



The steady-state neutron distribution in the system is characterized by $\psi(x, \mu)$, which is given by Eq. (13). It remains to determine the expansion coefficients from the boundary conditions

$$\psi(x, \mu) = \psi(-x, -\mu) \qquad (14)$$

and

$$\psi(-b, \mu > 0) = 0 \quad . \qquad (15)$$

Inserting Eq. (13) in (14) and using the symmetry properties of the eigenfunctions,

$$\psi_{0\pm}(-x, -\mu) = \psi_{0\mp}(x, \mu)$$

$$\psi_{\nu}(-x, -\mu) = \psi_{-\nu}(x, \mu) \quad ,$$

we find after some rearrangement that

$$(a_{0+} - a_{0-})\, \psi_{0+}(x, \mu) + (a_{0-} - a_{0+})\, \psi_{0-}(x, \mu) + \int_{-1}^{1} [A(\nu) - A(-\nu)]\, \psi_{\nu}(x, \mu)\, d\nu = 0 \quad . \qquad (14a)$$

Since this is to hold for all $|x| \leq b$ and all $|\mu| \leq 1$, we conclude that

$$a_{0+} = a_{0-} ; \qquad A(\nu) = A(-\nu) \quad . \qquad (14b)$$

Using the condition at x = -b, we obtain the following equation for the coefficients,

$$a_{0+} [\psi_{0+}(-b, \mu) + \psi_{0-}(-b, \mu)] + \int_{-1}^{1} A(\nu)\, \psi_{\nu}(-b, \mu)\, d\nu = 0 \quad , \quad \mu > 0 \quad . \qquad (15a)$$

It should be noted that Eq. (15a) is equally valid at x = b.

Equation (15a) can be put in a standard form for a nonhomogeneous singular integral equation by decomposing the integral term and using the fact that both $A(\nu)$ and $\lambda(\nu)$ are even functions:

$$A'(\mu)\lambda(\mu) + P\int_0^1 \frac{\left(\frac{c}{2}\right)\nu A'(\nu)}{\nu - \mu}\, d\nu = -a\left[\phi_{0+}(\mu) + \phi_{0-}(\mu)\, e^{-t/\nu_0}\right] - \int_0^1 \frac{\left(\frac{c}{2}\right)\nu A'(\nu)\, e^{-t/\nu}}{\nu + \mu}\, d\nu \quad , \quad \mu > 0 \tag{16}$$

where we have defined

$$A'(\nu) = A(\nu)\, e^{b/\nu}$$

$$a = a_{0+}\, e^{b/\nu_0} \quad . \tag{17}$$

The theory of reducing equations of the form of Eq. (16) to Fredholm equations is given by Muskhelishvili.(3) In essence, the procedure consists of rewriting Eq. (16) as

$$A'(\mu)\,\lambda(\mu) + P\int_0^1 \frac{\left(\frac{c}{2}\right)\nu A'(\nu)}{\nu - \mu}\, d\nu = \psi'(\mu) \tag{18}$$

with

$$\psi'(\mu) = -a\left[\phi_{0+}(\mu) + \phi_{0-}(\mu)\, e^{-t/\nu_0}\right] - \int_0^1 \frac{\left(\frac{c}{2}\right)\nu\, A'(\nu)\, e^{-t/\nu}}{\nu + \mu}\, d\nu \tag{18a}$$

and assuming temporarily that the $\psi'(\mu)$ is a known function. The unknown coefficient $A'(\nu)$ is then related to the boundary values of a sectionally holomorphic (analytic) function as it approaches the cut (0,1) from above and below. Finally, such a function is constructed from its properties and $A'(\nu)$ is determined from the boundary values of the constructed function. The discrete coefficient a (or a_{0+}) remains arbitrary, as indeed it should.

It is pointed out in Chapter 14 of Ref. 3 that the procedure of reducing singular to regular integral equations holds for sufficiently well-behaved functions $A'(\nu)$ and $\psi'(\mu)$. Specifically, we assume that both A' and ψ' satisfy the H* condition, i.e., they satisfy the Holder condition (H condition):

$$\left| A'(\mu_1) - A'(\mu_2)\right| \le M\left|\mu_1 - \mu_2\right|^{\alpha} \quad \text{with} \quad 0 < \alpha \le 1 \tag{19}$$

on the open interval $0 < \mu < 1$, with possible singularities at the ends $\beta = 0$ or 1, which are weaker however, than $1/\mu$;

$$A'(\mu) = \frac{A^*(\mu)}{|\mu - \beta|^\gamma} \quad \gamma < 1 \quad . \tag{19a}$$

In the above conditions, M is a positive constant and $A^*(\mu)$ is an arbitrary function which obeys the H condition (19) in the closed interval $0 \le \mu \le 1$.

Following Ref. 1 closely, we introduce the sectionally holomorphic function

$$N(z) = \left(\frac{1}{2\pi i}\right) \int_0^1 \frac{\left(\frac{c}{2}\right)\nu A'(\nu)}{\nu - z} d\nu \tag{20}$$

having the following properties:

a) N(z) is holomorphic in the plane cut from 0 to 1.

b) $N(z) \sim \frac{1}{z}$ as $z \to \infty$.

c) $N(z) < \frac{\text{const}}{|z - \beta|^\gamma} \quad \gamma < 1$ as $z \to \beta$.

The boundary values of N(z) as it approaches the line of discontinuity (0, 1) are given by the Plemelj formula[3]:

$$N^{\pm}(\mu) = \pm \frac{1}{2}\frac{c}{2}\mu A'(\mu) + \left(\frac{1}{2\pi i}\right) P \int_0^1 \frac{\left(\frac{c}{2}\right)\nu A'(\nu)}{\nu - \mu} d\nu \quad ,$$

from which it follows that

$$N^+ + N^- = \frac{1}{\pi i} P \int_0^1 \frac{\frac{c}{2} \nu A'(\nu)}{\nu - \mu} d\nu \tag{20a}$$

and

$$N^+ - N^- = \frac{c}{2}\mu A'(\mu) \quad . \tag{20b}$$

Substitution of Eq. (20a) and (20b) into (18) gives

$$G(\mu)\, N^{+}(\mu) - N^{-}(\mu) = \frac{\frac{c}{2}\mu\, \psi'(\mu)}{\lambda(\mu) - \left(\frac{c\pi i\mu}{2}\right)} \quad , \tag{21}$$

where we have defined

$$G(\mu) = \frac{\lambda(\mu) + \frac{c\pi i\mu}{2}}{\lambda(\mu) - \frac{c\pi i\mu}{2}} \quad . \tag{21a}$$

The reduction of Eq. (18) is now equivalent to the following boundary value problem: to find a sectionally holomorphic function N(z) having the aforementioned properties on the complex plane and subject to Eq. (21) on the boundary. This problem is referred to as the nonhomogeneous Hilbert problem and is discussed extensively in Ref. 3. Its solution involves the construction of a particular solution to the homogeneous Hilbert problem which results by equating the right side of (21) to zero. Let $[X(z)]^{-1}$ be that particular solution. By definition, it satisfies the properties of N(z) in the complex plane, and its boundary values are subject to the homogeneous part of Eq. (21):

$$X^{+}(\mu) = G(\mu)\, X^{-}(\mu) \quad . \tag{22}$$

The explicit form of X(z) can now be found by taking logarithms of Eq. (22), using the Plemelj formula, and requiring that the resulting expression has the correct behavior at the ends. From Table I of Ref. 1, the appropriate X(z) is

$$X(z) = \frac{1}{1 - z}\, e^{\Gamma(z)} \quad , \tag{23}$$

where

$$\Gamma(z) = \left(\frac{1}{2\pi i}\right) \int_0^1 \frac{\ln G(\mu)}{\mu - z}\, d\mu \quad . \tag{23a}$$

Inserting $G(\mu)$ from Eq. (22) in Eq. (21), we obtain

$$X^{+} N^{+} - X^{-} N^{-} = \gamma(\mu)\, \psi'(\mu) \quad , \tag{24}$$

where

$$\gamma(\mu) = \frac{\left(\frac{c}{2}\right) \mu X^{-}(\mu)}{\lambda(\mu) - \left(\frac{c\pi i\mu}{2}\right)} \tag{24a}$$

The solution of the nonhomogeneous Hilbert problem now follows immediately from (24) and the use of Plemelj's formula:

$$N(z) = \frac{1}{2\pi i\, X(z)} \int_0^1 \frac{\gamma(\mu)\, \psi'(\mu)}{\mu - z}\, d\mu + \frac{P_k(z)}{X(z)} \quad , \tag{25}$$

where $P_k(z)$ is an arbitrary polynomial of degree k. Since $X(z) \sim 1/z$ at infinity, it follows from Eq. (25) that N(z) will have this behavior only if $P_k(z) \equiv 0$ and

$$\int_0^1 \gamma(\mu)\, \psi'(\mu)\, d\mu = 0 \quad . \tag{26}$$

To complete the reduction of Eq. (18), it is now only necessary to find $N^{\pm}(\mu)$ from Eq. (25) and use Eq. (20b). The result is

$$A'(\mu) = \lambda(\mu)\, g(c,\mu)\, \psi'(\mu) - \frac{1}{X^-(\mu)\left[\lambda(\mu) + \frac{c\pi i\mu}{2}\right]} P \int_0^1 \frac{\gamma(\nu)\, \psi'(\nu)}{\nu - \mu}\, d\nu \quad , \tag{27}$$

where

$$g(c,\mu) = \frac{1}{\lambda^2(\mu) + \left(\frac{c^2\pi^2\mu^2}{4}\right)} \quad . \tag{27a}$$

The proposed integral equation for the coefficients now follows by inserting Eq. (18a) for $\psi'(\mu)$. The final form of this equation, as well as many subsequent results is, however, considerably simplified by two identities for the function X(z). These identities were first proved by Case.(4) The proofs are sketched in Appendix A.

1. $$X(z) = \int_0^1 \frac{\gamma(\mu)\, d\mu}{\mu - z}$$

2. $$X(z)\, X(-z) = \frac{1 - c z \tanh^{-1} z}{(\nu_0^2 - z^2)\, (1 - c)} \quad .$$

The discrete terms of Eq. (18a) give rise to integrals of the form

$$\int_0^1 \frac{\phi_{0\pm}(\nu)\, \gamma(\nu)\, d\nu}{\nu - \mu} = \frac{c}{2}\nu_0 \int_0^1 \frac{\gamma(\nu)\, d\nu}{(\nu_0 \mp \nu)(\nu - \mu)} \quad .$$

Decomposing the denominator by partial fractions and using the first identity, we find

$$\int_0^1 \frac{\phi_{0\pm}(\nu)\,\gamma(\nu)d\nu}{\nu - \mu} = \frac{c}{2}\,\nu_0\,\frac{1}{\nu_0 \mp \mu}\,[X(\mu) - X(\pm\nu_0)] \quad .$$

Similarly, the integral term of Eq. (18a) gives

$$\int_0^1 \frac{\gamma(\nu)\,d\nu}{\nu - \mu}\int_0^1 \frac{\frac{c}{2}\alpha\, A'(\alpha)\, e^{-t/\alpha}}{\alpha + \nu}\, d\alpha = \int_0^1 \frac{\frac{c}{2}\alpha\, A'(\alpha)\, e^{-t/\alpha}}{\alpha + \mu}[X(\mu) - X(-\alpha)]\, d\alpha \quad .$$

Finally, from the second identity, we see that

$$\frac{1}{X^-(\mu)\left[\lambda(\mu) + \frac{c\pi i \mu}{2}\right]} = g\,(c,\mu)\,(\nu_0^2 - \mu^2)\,(1 - c)\,X(-\mu) \quad .$$

Substitution in Eq. (27) and cancellation of common terms gives

$$\begin{aligned} A'(\mu) = &-a\left[\phi_{0+}(\mu)\,X(\nu_0) + \phi_{0-}(\mu)\,X(-\nu_0)\,e^{-t/\nu_0}\right] \\ &\times (\nu_0^2 - \mu^2)\,(1 - c)\,X(-\mu)\,g(c,\mu) \\ &- (\nu_0^2 - \mu^2)\,(1 - c)\,g(c,\mu)\int_0^1 \frac{\frac{c}{2}\,\nu\, X(-\nu)\, A'(\nu)\, e^{-t/\nu}}{\nu + \mu}\, d\nu \quad . \end{aligned} \tag{28}$$

By a similar procedure, the auxiliary condition (26) can also be put in a more explicit form:

$$a\,\frac{c\nu_0}{2}\,[X(\nu_0) - e^{-t/\nu_0}\,X(-\nu_0)] = \int_0^1 \frac{c}{2}\,\nu\, X(-\nu)\, A'(\nu)\, e^{-t/\nu}\, d\nu \quad . \tag{29}$$

Equation (29) states that a solution of the transport equation with the given boundary conditions exists only if there is a definite relation between material concentrations (given by c or ν_0) and the size given by t. This, therefore, corresponds to an exact statement of the criticality condition.

To summarize, we have shown that the angular neutron density in a one-dimensional multiplying medium can be exactly represented by

$$\psi(x,\mu) = a_{0+}\,[\psi_{0+}(x,\mu) + \psi_{0-}(x,\mu)] + \int_{-1}^{1} A(\nu)\,\psi_\nu(x,\mu)\, d\nu \quad , \tag{30}$$

where a_{0+} is an arbitrary constant and $A(\nu)$ obeys the Fredholm integral equation (28). The criticality condition is given by Eq. (29).

From Eq. (3) and the normalization condition the neutron density becomes

$$\rho(x) = a_{0+} [e^{-x/\nu_0} + e^{x/\nu_0}] + \int_{-1}^{1} A(\nu)\, e^{-x/\nu}\, d\nu \quad . \tag{31}$$

This expression can be put in a more recognizable form by recalling that ν_0 is purely imaginary and $A(\nu)$ is an even function:

$$\rho(x) = 2a_{0+} \cos (x/|\nu_0|) + 2 \int_{0}^{1} A(\nu) \cosh (x/\nu)\, d\nu \quad , \tag{32}$$

where $\nu_0 = i|\nu_0|$. The first term of (32) is immediately identified with the asymptotic solution from diffusion theory. Also, $\rho(x)$ possesses the expected symmetry about the origin.

A similar procedure gives the following expression for the neutron current:

$$j(x) = 2a_{0+}(c - 1)\, |\nu_0| \sin (x/|\nu_0|) + 2(c - 1) \int_{0}^{1} A(\nu)\, \nu \sinh (x/\nu)\, d\nu \quad . \tag{33}$$

As expected, j vanishes at the origin. A further check is obtained by differentiating Eq. (33) with respect to x. This yields the continuity equation

$$\frac{dj(x)}{dx} + (1 - c)\, \rho(x) = 0 \quad , \tag{34}$$

which could also be obtained by integrating the original equation.

IV. APPROXIMATE SOLUTIONS

It is clear from the results of the previous section that an explicit solution of Eq. (2) is not possible. The major advantage of this approach is that it affords a systematic approximation method by which the desired results can be computed to any accuracy. Moreover, a number of very interesting results emerge from the approximations.

The approximation procedure which will be used in this section involves keeping successive terms of the Neumann series solution[6] of Eq. (28). It is shown in Appendix B that such a series converges very rapidly for values of $1 \leq c < 2$, which are those of practical interest.

We start by writing Eq. (28) as

$$A'(\nu) = A_1'(\nu) + \lambda \int_0^1 K(\nu,\alpha)\, A'(\alpha)\, d\alpha \quad , \tag{35}$$

where $A_1'(\nu)$ denotes the free term of (28),

$$\lambda = \frac{c}{2}(1 - c) \tag{35a}$$

and

$$K(\nu,\alpha) = -\frac{(\nu_0^2 - \nu^2)\, X(-\nu)\, g(c,\nu)\, X(-\alpha)\, \alpha e^{-t/\alpha}}{\nu + \alpha} \tag{35b}$$

is the kernel. The Neumann series of Eq. (35) has the form

$$A'(\nu) = A_1'(\nu) + \lambda\phi_1(\nu) + \lambda^2\phi_2(\nu) + \cdots \quad , \tag{36}$$

where

$$\phi_n(\nu) = \int_0^1 K_n(\nu,\alpha)\, A_1'(\alpha)\, d\alpha \tag{36a}$$

and

$$K_n(\nu,\alpha) = \int_0^1 K(\nu,\beta)\, K_{n-1}(\beta,\alpha)\, d\beta \quad . \tag{36b}$$

For the zeroth approximation, we take $A'(\nu) = 0$. Since this is strictly true for $c = 1$ (Appendix B), the results would be expected to hold for large systems.

The criticality condition in this case becomes

$$\frac{X(\nu_0)}{X(-\nu_0)} - e^{-t/\nu_0} = 0 \quad . \tag{37}$$

The ratio $X(-\nu_0)/X(\nu_0)$ is simply related to the Milne problem extrapolation distance $z_0(c)$ (Appendix A):

$$\frac{X(-\nu_0)}{X(\nu_0)} = -e^{-2z_0/\nu_0} \quad . \tag{38}$$

Combining Eqs. (37) and (38), we get

$$e^{2z_0/\nu_0} + e^{-t_0/\nu_0} = 0$$

or

$$\cos(t_0/2|\nu_0| + z_0/|\nu_0|) = 0 \quad , \tag{39}$$

from which emerges the familiar result

$$t_0 = \pi|\nu_0| - 2z_0 \quad , \tag{40}$$

where t_0 is the thickness in mean free paths for $A'(\mu) = 0$.

This approximation also provides the means of testing the validity of the commonly used boundary condition of no re-entrant current. If we let $A(\nu) = 0$ in Eq. (13), and replace the exact boundary condition at the surface by

$$j_+(-b) = \int_0^1 \mu\psi(-b,\mu)\, d\mu = 0 \quad , \tag{41}$$

we obtain, instead of (40),

$$t_0' = \pi|\nu_0| - 2z_0' \quad , \tag{41a}$$

where

$$z_0' = |\nu_0| \tan^{-1} \frac{c - 1}{c|\nu_0| \ln\sqrt{1 + 1/|\nu_0|^2}} \quad . \tag{41b}$$

For $(c-1) << 1$, Eq. (41b) reduces to

$$z_0' \simeq \frac{1}{\sqrt{3(c-1)}} \tan^{-1} 2\left(\frac{c-1}{3}\right)^{1/2}$$

$$\simeq \frac{2}{3}\left[1 + \frac{4}{9}(1-c) + \cdots\right] \quad . \tag{41c}$$

Equation (41c) is identical with the extrapolation distance in the P_1 approximation.[7] Thus, the replacement of the exact boundary condition by Eq. (41) consists of replacing the exact Milne problem extrapolation distance by z_0' of Eq. (41b).

The neutron distribution in this approximation is described by the following expressions:

a) The angular density

$$\psi_0(x,\mu) = a_{0+}\left[\phi_{0+}(\mu)\, e^{-x/\nu_0} + \phi_{0-}(\mu)\, e^{x/\nu_0}\right]$$

$$= \frac{a_{0+}\, c|\nu_0|}{|\nu_0|^2 + \mu^2}\left[|\nu_0| \cos(x/|\nu_0|) + \mu \sin(x/|\nu_0|)\right] \quad . \tag{42}$$

This expression clearly possesses the required symmetry in x and μ.

b) The neutron density

$$\rho_0(x) = 2\,a_{0+} \cos\left(x/|\nu_0|\right) \quad . \tag{43}$$

c) The neutron current

$$j_0(x) = 2\,a_{0+} |\nu_0|\,(c - 1) \sin\left(x/|\nu_0|\right) \quad . \tag{44}$$

A comparison of Eqs. (43) and (44) gives

$$j_0(x) = -D \frac{d\rho_0(x)}{dx} \tag{44a}$$

where

$$D = |\nu_0|^2 (c - 1) \quad . \tag{44b}$$

This definition for the diffusion coefficient D degenerates to the familiar result of $D = \frac{1}{3}$ in units of mean free path for c close to unity. Moreover, combining Eqs. (44a) and (44b) with the continuity equation (34), we obtain the standard diffusion equation

$$\frac{d^2\rho_0(x)}{dx^2} + \frac{1}{|\nu_0|^2}\,\rho_0(x) = 0 \quad , \tag{44c}$$

where $1/|\nu_0|$ plays the role of the "buckling."

For the first approximation we let $A'(\nu)$ equal the first term of Eq. (36):

$$A'(\nu) = A_1'(\nu) = \frac{a\,c\,\nu_0}{2}(c - 1)\,X(-\nu)\,g(c,\nu)$$
$$\times \left[\nu_0\left(X(\nu_0) + X(-\nu_0)\,e^{-t/\nu_0}\right) + \nu\left(X(\nu_0) - X(-\nu_0)\,e^{-t/\nu_0}\right)\right] \quad . \tag{45}$$

Inserting this in the criticality statement, we find

$$X(\nu_0) - X(-\nu_0)\,e^{-t/\nu_0} = \frac{c}{2}(1 - c)\left\{\nu_0\left[X(\nu_0) + X(-\nu_0)\,e^{-t/\nu_0}\right] I_1 + \left[X(\nu_0) - X(-\nu_0)\,e^{-t/\nu_0}\right] I_2\right\} \quad , \tag{46}$$

where

$$I_1 = \int_0^1 \nu\,X^2(-\nu)\,g(c,\nu)\,e^{-t/\nu}\,d\nu \tag{46a}$$

and

$$I_2 = \int_0^1 \nu^2 X^2(-\nu)\, g(c,\nu)\, e^{-t/\nu}\, d\nu \quad . \tag{46b}$$

A rearrangement of Eq. (46) gives

$$\frac{X(\nu_0) - X(-\nu_0)\, e^{-t/\nu_0}}{X(\nu_0) + X(-\nu_0)\, e^{-t/\nu_0}} = \nu_0\, \alpha(c,t) \quad , \tag{46c}$$

where the function $\alpha(c,t)$ is defined by

$$\alpha = \frac{\frac{c}{2}(c-1)\, I_1}{1 - \left(\frac{c}{2}\right)(c-1)\, I_2} \quad . \tag{46d}$$

With the aid of Eq. (38), we finally arrive at the criticality condition corrected to first order:

$$\tan \frac{1}{2|\nu_0|}\left[t_1 + 2z_0 - \pi\,|\nu_0|\right] = -|\nu_0|\,\alpha \quad . \tag{47}$$

Further insight is gained by examining this result for small values of $\alpha|\nu_0|$. Then,

$$\begin{aligned} t_1 &\simeq \pi\,|\nu_0| - 2z_0 - 2\alpha(c,t)\,|\nu_0|^2 \\ &\simeq t_0 - 2|\nu_0|^2\,\alpha(c,t) \quad , \end{aligned} \tag{48}$$

where t_0 is the zeroth-order thickness. Since $\alpha(c,t)$ is positive, it follows from Eq. (48) that the transport correction introduced by the first-order approximation has the effect of decreasing the critical thickness. For the cases where the argument $t_1 - t_0$ is not small, the transcendental equation (47) can of course be solved graphically.

Before presenting the expressions for ψ, $\rho(x)$, and $j(x)$ in this approximation, it is convenient to simplify the form of $A_1'(\mu)$ by judicial grouping of constants. First, we note that the quantity $X(\nu_0)\, e^{t/\nu_0}/X(-\nu_0)$ can be expressed in terms of $\alpha(c,t)$ with the aid of (46c)

$$\frac{X(\nu_0)}{X(-\nu_0)}\, e^{t/\nu_0} = \frac{1 + \alpha\,\nu_0}{1 - \alpha\,\nu_0} \quad ,$$

from which it follows that

$$X(-\nu_0)\, e^{-b/\nu_0} = \left[X(\nu_0)\, X(-\nu_0)\, \frac{1 - \alpha\,\nu_0}{1 + \alpha\,\nu_0}\right]^{1/2} \quad .$$

Recalling the definitions of a and $A'(\mu)$ from Eq. (17), we obtain

$$A_1(\mu) = a_{0+} B(1 + \alpha\mu) X(-\mu) g(c,\mu) e^{-b/\mu} \quad , \tag{49}$$

where

$$B = c\nu_0^2 (c - 1)\left[\frac{X(\nu_0) X(-\nu_0)}{1 - \alpha^2 \nu_0^2}\right]^{1/2} \quad . \tag{49a}$$

The neutron distribution is now given by the following expressions:

a) Angular density

$$\psi_1(x,\mu) = \psi_0(x,\mu) - q(x,\mu) \quad , \tag{50}$$

where $\psi_0(x,\mu)$ is given by Eq. (42) and

$$q(x,\mu) = -A_1(\mu)\lambda(\mu) e^{\mp x/\mu} - P\int_0^1 \frac{c}{2} A_1(\nu)\,\nu\left[\frac{e^{-x/\nu}}{\nu - \mu} + \frac{e^{x/\nu}}{\nu + \mu}\right] d\nu \tag{50a}$$

is a positive quantity representing a first-order transport correction. The upper and lower signs in (50a) refer to positive and negative values of μ, respectively.

b) Neutron density

$$\rho_1(x) = \rho_0(x) - h(x) \quad , \tag{51}$$

where

$$h(x) = -2a_{0+} B\int_0^1 X(-\nu) g(c,\nu)(1 + \alpha\nu) e^{-b/\nu} \cosh(x/\nu)\, d\nu \tag{51a}$$

is a positive correction to the asymptotic density.

c) Neutron current

$$j_1(x) = j_0(x) - 2a_{0+}(1 - c) B\int_0^1 \nu X(-\nu) g(c,\nu)(1 + \alpha\nu) e^{-b/\nu} \sinh(x/\nu)\, d\nu \quad , \tag{52}$$

where the correction term is again positive.

Approximations of higher order become too unwieldy for hand computations and do not seem to add further analytical insight to the neutron distribution or criticality condition. In addition, it is shown in Appendix B that the contribution of such terms is negligible except for systems with dimensions less than one mean free path.

V. NUMERICAL RESULTS

The magnitude of the first-order correction terms was computed, first, in order to examine their contribution and, secondly, to demonstrate the applicability of the approximation procedure. Calculations for the neutron distribution were made by means of the normalization

$$\rho_0(0) = 2a_{0+} = 1 \quad . \tag{53}$$

The various integrals were evaluated with a four-point Gaussian quadrature formula[8] which corresponded to an approximation of the integrands by a seventh-order polynomial. The numerical results for $X(-\nu)$, the critical thickness, $\rho(x)$ and $\psi(x,\mu)$ for various values of c will be given in order.

Computations of the function $X(-\nu)$ are facilitated by two additional identities. These identities are also due to Case and are proved in Appendix A.

$$3. \quad X(-\nu) = \exp\left[-\frac{c}{2}\int_0^1 g(c,\mu)\left(1+\frac{c\mu^2}{1-\mu^2}\right)\ell n\,(\mu+\nu)\,d\mu\right]$$

$$4. \quad X(-\nu) = \frac{c}{2}\frac{1}{1-c}\int_{-1}^{0}\frac{\mu\,d\mu}{(\nu_0^2-\mu^2)\,X(\mu)\,(\mu-\nu)} \quad , \quad \nu > 0 \quad .$$

From the second identity given previously on page 11 it follows that

$$X(0) = \sqrt{\frac{1}{\nu_0^2\,(1-c)}} \tag{54}$$

and

$$X(\nu_0)\,X(-\nu_0) = \frac{1}{2}\left[\frac{X^2(0)-1}{1-\nu_0^2}\right] \quad . \tag{55}$$

The third identity, given above, was used for numerical integrations. The results for five values of c are shown in Fig. 1. A comparison of $X(0)$ from the numerical integration and the exact value given by Eq. (54) shows a deviation of 1 per cent for c = 1.01 and 3 per cent at c = 2.00.

An approximate expression for $X(-\nu)$ can be obtained by noting that the quantity $g(c,\mu)[1+(c\mu^2)/(1-\mu^2)]$ which appears in the integrand of the third identity is slowly varying except near $\mu = 1$. Taking this quantity as unity, the third identity reduces to

$$X(-\nu) \simeq X(0)\left(\frac{1+\nu}{\nu}\right)^{-c\nu/2}(\nu+1)^{-c/2} \quad , \quad \nu > 0 \quad . \tag{56}$$

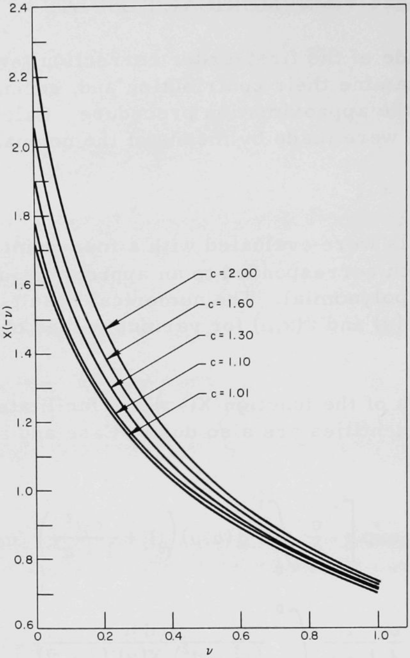


Fig. 1. The Function $X(-\nu)$ for Various Values of c

As would be expected, the largest deviation of this expression from the values of Fig. 1 occurs for $\nu = 1$. For $c = 1.01$, Eq. (56) overestimates the values from Fig. 1 by 18.5 per cent while for $c = 2.00$ it gives an underestimate of ~21 per cent. Another approximate expression for $X(-\nu)$ can be obtained from the fourth identity. From Fig. 1, $X(-\nu)$ varies roughly as

$$X(-\nu) \sim \frac{X(0)}{1+\nu} \quad , \quad \nu > 0 \quad . \tag{57}$$

Using (57) as a trial function in the integrand of (4) we find

$$X(-\nu) \simeq X(0) \left\{ 1 - \frac{c\nu|\nu_0|^2}{2(|\nu_0|^2 + \nu^2)} \left[(1-\nu) \ln \frac{1+\nu}{\nu} + \frac{|\nu_0|^2 + \nu}{c|\nu_0|^2} - \frac{1-\nu}{2} \ln \frac{|\nu_0|^2 + 1}{|\nu_0|} \right] \right\} \tag{58}$$

This expression represents $X(-\nu)$ quite accurately near $\nu = 0$. However for $\nu = 1$, it becomes

$$X(-1) = \frac{X(0)}{2} \quad ,$$

which is the same as the trial function.

The critical thickness t was determined by first evaluating $\alpha(c,t)$ and then solving Eq. (47) graphically. The results for five values of c are given in Table I.

Table I

CRITICAL THICKNESS AS COMPUTED FROM
EQS. (40), (47) and (60)

c	t_1	t_1'	t_0	% error $= \frac{t_0 - t_1}{t_1} \times 100$
1.01	16.69	16.69	16.69	$\sim 6 \times 10^{-11}$
1.10	4.24	4.24	4.24	$\sim 4 \times 10^{-4}$
1.30	1.780	1.780	1.783	0.170
1.60	1.022	1.025	1.030	0.775
2.00	0.621	0.625	0.634	2.05

Since $\alpha(c,t)$ is small even for $c = 2.00$, it is possible to estimate the correction term in Eq. (48) without introducing an appreciable error in t_1. Noting that $I_2 \ll I_1$, we can rewrite (46d) as

$$\alpha(c,t) \simeq \frac{c}{2}(c - 1)\, I_1 \quad . \tag{59}$$

An examination of the integrand of I_1 shows that the greatest contribution to this integral comes from the neighborhood of the upper limit. Also, near $\nu = 1$,

$$X(-\nu) \simeq \frac{X(0)}{2} \quad ; \quad g(c, \nu) \simeq \left(\frac{2}{c\pi}\right)^2 \quad .$$

Consequently,

$$I_1 \simeq \frac{X^2(0)}{4}\left(\frac{2}{c\pi}\right)^2 E_3(t) \quad , \tag{59a}$$

where $E_n(x)$ is the exponential integral, and from Eq. (59)

$$\alpha(c,t) \simeq \frac{(c-1)\,X^2(0)}{2\pi^2 c}\,E_3(t) \quad . \tag{59b}$$

Substitution of (59b) into (48) and use of (54) gives

$$t_1' = t_0 - \frac{E_3(t_1')}{c\,\pi^2} \quad . \tag{60}$$

The values of t_1' from Eq. (60) are also given in Table I for comparison. In all cases, t is in units of mean free paths. The conclusion of major interest derived from these results is that even for the extreme case of c = 2.00 the first-order transport corrections for t is of the order of only 2 per cent.

The numerical results for the neutron density are summarized in Table II. A plot of $\rho(x)$ vs x for three values of c is given in Figs. 2 and 3. In Table II we have also tabulated the ratio

$$\frac{\rho_0(x)}{\rho_1(x)} = 1 + \frac{h(x)}{\rho_1(x)} \quad , \tag{61}$$

which serves as a measure of the contribution of the first-order transport correction to the asymptotic density.

Table II

NEUTRON DENSITY AS A FUNCTION OF POSITION

x/b	c = 1.01			c = 1.10			c = 1.30			c = 1.60			c = 2.00		
	ρ_0	ρ_1	ρ_0/ρ_1	ρ_0	ρ_1	ρ_0/ρ_1	ρ_0	ρ_1	ρ_0/ρ_1	ρ_0	ρ_1	ρ_0/ρ_1	ρ_0	ρ_1	ρ_0/ρ_1
0	1.000	1.000	1.000	1.000	0.9978	1.002	1.000	0.9804	1.020	1.000	0.9525	1.050	1.000	0.9247	1.082
0.25	0.9345	0.9345	1.000	0.9547	0.9517	1.003	0.9742	0.9490	1.023	0.9783	0.9208	1.054	0.9837	0.9054	1.086
0.50	0.7485	0.7485	1.000	0.8231	0.8170	1.007	0.8842	0.8556	1.032	0.9143	0.8518	1.073	0.9352	0.8468	1.104
0.75	0.4722	0.4718	1.001	0.6177	0.6018	1.026	0.7453	0.7008	1.064	0.8104	0.7309	1.109	0.8562	0.7417	1.154
0.85	0.3342	0.3332	1.003	0.5217	0.4972	1.049	0.6774	0.6202	1.094	0.7596	0.6628	1.144	0.8166	0.6836	1.195
0.95	0.1985	0.1916	1.036	0.4130	0.3689	1.120	0.6022	0.5183	1.1619	0.7015	0.5700	1.230	0.7729	0.6113	1.264
1.00	0.1215	0.0969	1.254	0.3577	0.2844	1.1257	0.5640	0.4430	1.271	0.6712	0.5202	1.289	0.7495	0.5706	1.313

For c close to unity, a procedure similar to that leading to Eq. (60) gives for h(x)

$$h(x) \simeq -\frac{BX(0)}{4}\left[E_2(b-x) + E_2(b+x)\right] \quad . \tag{62}$$

For c = 1.10, the maximum difference in the values of $\rho_1(x)$ from Table II and from Eq. (62) is about 2.4 per cent. Since the constant B is negative, h(x) is positive and increases with c and x. The maximum correction to $\rho_0(x)$ therefore occurs at the boundary, which, of course, was anticipated.

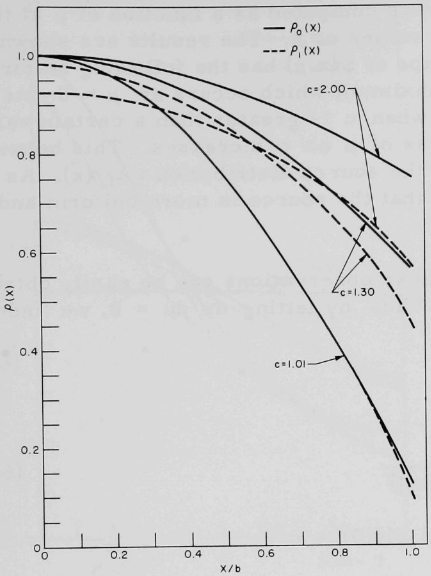


Fig. 2

Neutron Density as a Function of Position

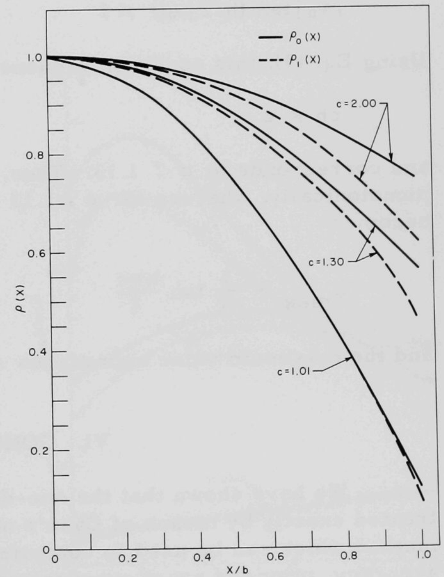


Fig. 3

Neutron Density Normalized so that $\rho_0(0) = \rho_1(0) = 1$

Finally, the angular density was computed as a function of μ at the origin and at the boundary for three values of c. The results are shown in Figs. 4 and 5. Qualitatively, the shape of $\psi(x,\mu)$ has the following features. At the origin, $\psi(0,\mu)$ always has a maximum which occurs at $\mu = 0$. At the boundary, a maximum appears only when c is greater than a certain value and shifts from higher to lower values of μ as c increases. This behavior is explained by the nonuniformity of the source distribution $\nu\Sigma_f \rho(x)$. As c increases, $\rho(0) - \rho(b)$ decreases, so that the source is more uniform and the peak value moves towards $\mu = 0$.

Quantitative verification of these observations can be easily obtained in the zeroth-order approximation. Thus, by letting $d\psi/d\mu = 0$, we find

$$\mu_{max} = |\nu_0| \tan (x/2|\nu_0|) \tag{63}$$

and

$$\psi_0(x,\mu_{max}) = \frac{c}{2} \frac{1}{1 + \tan^2 (x/2|\nu_0|)} \quad . \tag{64}$$

For $x = 0$, $\quad \mu_{max} = 0$ and $\psi(0,0) = c/2$.

For $x = b$, a maximum exists only if

$$|\nu_0| \tan (b/2|\nu_0|) \leq 1 \quad . \tag{64a}$$

Using Eq. (9), this condition reduces to

$$cb \leq 2$$

and corresponds to $c \simeq 1.13$. Thus, for $c < 1.13$, $\psi_0(b,\mu)$ increases monotonically, whereas for $c \gtrsim 1.13$ it has a maximum. For $c >> 1$, Eq. (64) becomes

$$\mu_{max} \sim \frac{2}{\pi c} \tan \frac{b\pi c}{4} \quad ,$$

and the maximum value approaches $c/2$.

VI. CONCLUSIONS

We have shown that the one-dimensional critical problem can be treated exactly by means of Case's normal mode expansion method. The method which can be used to compute these quantities to any desirable accuracy, provides exact expressions for the neutron distribution and criticality condition.

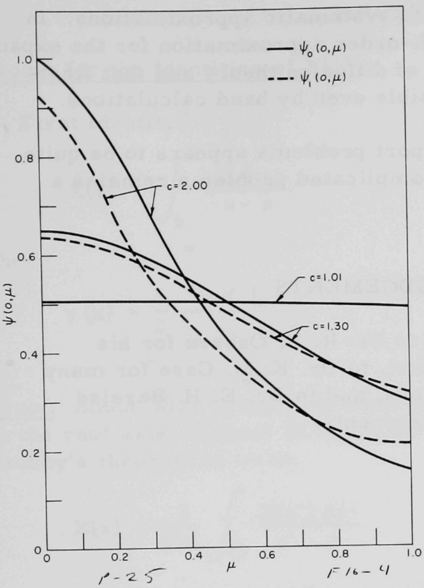


Fig. 4

Angular Distribution at the Origin

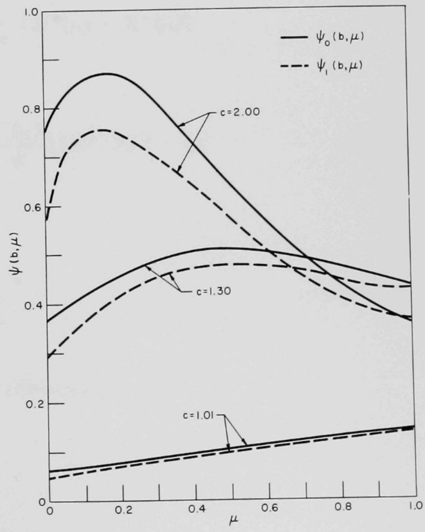


Fig. 5

Angular Distribution at the Boundary

Moreover, the results are adaptable to systematic approximations. In particular, it was shown that a zeroth-order approximation for the expansion coefficients leads to the results of diffusion theory and that first-order transport corrections are possible even by hand calculations.

This method of treating transport problems appears to be quite powerful. Its applicability to more complicated problems remains a question for investigation.

ACKNOWLEDGEMENTS

The author is deeply indebted to Dr. R. K. Osborn for his constant supervision and encouragement, to Dr. K. M. Case for many enlightening discussions and suggestions, and to Dr. E. H. Bareiss for his thorough review and helpful comments.

APPENDIX A

Proofs for the Identities of X(z)

1. First Identity:

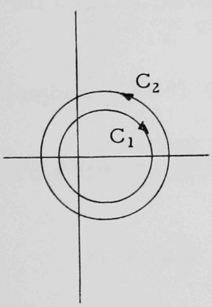
$$X(z) = \int_0^1 \frac{\gamma(\mu)\, d\mu}{\mu - z} \quad ,$$

where

$$\gamma(\mu) = \frac{c}{2} \frac{\mu X^-(\mu)}{\lambda(\mu) - \frac{c\pi i \mu}{2}} \quad .$$

Proof. Since X(z) is analytic over the whole plane with a branch cut on the real axis segment (0, 1) and behaves as 1/z at infinity, we can use Cauchy's theorem to write

$$X(z) = \frac{1}{2\pi i} \int_C \frac{X(z')\, dz'}{z' - z} \tag{A.1}$$



where the contour C consists of two parts, as shown in the accompanying figure. If C_2 is taken at infinity and C_1 around the cut, Eq. (A.1) reduces to

$$X(z) = \frac{1}{2\pi i} \int_0^1 \frac{d\mu}{\mu - z} [X^+(\mu) - X^-(\mu)]$$

$$= \frac{1}{2\pi i} \int_0^1 \frac{d\mu X^-(\mu)}{\mu - z} [(X^+/X^-) - 1] \quad . \tag{A.2}$$

But from Eq. (22) we have

$$\frac{X^+}{X^-} - 1 = G(\mu) - 1 = \frac{c\pi i \mu}{\lambda(\mu) - \frac{c\pi i\mu}{2}} \quad . \tag{A.3}$$

Inserting (A.3) into (A.2) gives the first identity.

2. Second Identity:

$$X(z)\, X(-z) = \frac{1 - c\, z \tanh^{-1} z}{(\nu_0^2 - z^2)(1 - c)}$$

Proof: This identity was proved by Case by using the properties of X(z). Here we show that it also follows from the definition of X(z);

$$X(z) = \frac{1}{1-z} e^{\Gamma(z)} \quad . \tag{23}$$

Consequently,

$$X(z)\, X(-z) = \frac{1}{1-z^2} e^{\Gamma(z) + \Gamma(-z)} \quad . \tag{A.4}$$

But from Eq. (23a), we can write

$$\Gamma(z) + \Gamma(-z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\ln G(\mu)}{\mu - z}\, d\mu \quad . \tag{A.5}$$

The integral in (A.5) was evaluated in Appendix A of Ref. (1):

$$\Gamma(z) + \Gamma(-z) = \ln \left[\frac{1 - z^2}{\nu_0^2 - z^2} \; \frac{1 - c z \tanh^{-1} z}{1 - c} \right] \quad . \tag{A.6}$$

Substitution of (A.6) in (A.4) completes the proof. Moreover, at z = 0 the identity reduces to

$$X^2(0) = \frac{1}{\nu_0^2 (1 - c)} \quad , \tag{A.7}$$

from which we obtain Eq. (54). Also, by taking the limit as $z \to \nu_0$, we find

$$X(\nu_0)\, X(-\nu_0) = \frac{1}{2} \frac{1}{\nu_0^2 (1 - c)} \left[\frac{1 - \nu_0^2 (1 - c)}{1 - \nu_0^2} \right]$$

$$= \frac{1}{2} \frac{X^2(0) - 1}{1 - \nu_0^2} \quad . \tag{55}$$

3. Third Identity:

$$X(z) = \exp \left[-\frac{c}{2} \int_{0}^{1} g(c, \mu) \left(1 + \frac{c\mu^2}{1 - \mu^2} \right) \ln (\mu - z)\, d\mu \right] \quad ,$$

where

$$g(c, \mu) = \frac{1}{\lambda^2(\mu) + \frac{c^2 \pi^2 \mu^2}{4}}$$

Proof: In Eq. (23) let

$$G(\mu) = e^{2i\theta(\mu)}$$

where

$$\theta(\mu) = \arg\left[\lambda(\mu) + \frac{c\pi i\mu}{2}\right] \tag{A.8}$$

and $\theta(0) = 0$, $\theta(1) = \pi$. Then

$$\Gamma(z) = \frac{1}{\pi}\int_0^1 \frac{\theta(\mu)\, d\mu}{\mu - z} \quad .$$

Integration by parts gives

$$\Gamma(z) \doteq \ln(1 - z) - \frac{1}{\pi}\int_0^1 \frac{d\theta}{d\mu} \ln(\mu - z)\, d\mu \quad . \tag{A.9}$$

But

$$\frac{d\theta}{d\mu} = \frac{c\pi}{2}\, g(c,\mu)\left(1 + \frac{c\mu^2}{1 - \mu^2}\right) \quad . \tag{A.10}$$

Combining (A.9), (A.10), and Eq. (23), we obtain the third identity.

A similar procedure can be used to relate the ratio $X(-\nu_0)/X(\nu_0)$ to the Milne problem extrapolation distance:

$$z_0 = \nu_0 \frac{c}{2}\int_0^1 g(c,\mu)\left(1 + \frac{c\mu^2}{1 - \mu^2}\right) \tanh^{-1}\left(\frac{\mu}{\nu_0}\right) d\mu \quad . \tag{A.11}$$

From equations (23) and (A.8), we have

$$\frac{X(-\nu_0)}{X(\nu_0)} = \frac{1 - \nu_0}{1 + \nu_0} \exp\left[-\frac{2\nu_0}{\pi}\int_0^1 \frac{\theta(\mu)}{\mu^2 - \nu_0^2}\, d\mu\right]$$

$$= \frac{1 - \nu_0}{1 + \nu_0} \exp\left[\frac{2}{\pi}\int_0^1 \theta(\mu)\left(\frac{d}{d\mu}\tanh^{-1}\frac{\mu}{\nu_0}\right) d\mu\right] \quad . \tag{A.12}$$

Integrating by parts and using (A, 10) for $d\theta(\mu)/d\mu$ gives

$$\frac{X(-\nu_0)}{X(\nu_0)} = \frac{1 - \nu_0}{1 + \nu_0} \exp\left[2\tanh^{-1}(1/\nu_0) - (2 z_0/\nu_0)\right]$$

$$= -e^{-2z_0/\nu_0} \quad . \tag{A.13}$$

4. Fourth Identity:

$$X(-\mu) = \frac{c}{2} \frac{1}{1-c} \int_{-1}^{0} \frac{\mu' \, d\mu'}{(\nu_0^2 - \mu'^2)\, X(\mu')(\mu' - \mu)} \quad . \quad \mu > 0 \quad .$$

Proof: This integral equation follows directly from the first two identities. From the second identity we find

$$\frac{X^-(\mu)}{\lambda(\mu) - \frac{c\pi i \mu}{2}} = \frac{1}{1-c} \frac{1}{\nu_0^2 - \mu^2} \frac{1}{X(-\mu)} \quad . \tag{A.14}$$

Substitution of (A.14) into the first identity and changing μ to μ' gives the integral equations.

APPENDIX B

Approximations for the Coefficients $A'(\nu)$

1. Convergence of the Neumann Series:

We first investigate the convergence of the series (36). The discussion follows the method given in Tricomi,[6] since the kernel $K(\nu,\alpha)$ clearly satisfies the required property of square integrability. The parameter λ is chosen as

$$\lambda = \frac{c}{2}(1 - c) \quad . \tag{B.1}$$

The general solution of Eq. (35) is

$$A'(\nu) = A_1'(\nu) + \sum_{n=1}^{\infty} \lambda^n \phi_n(\nu)$$

$$= A_1'(\nu) + \lambda \sum_{n=1}^{\infty} \lambda^{n-1} \int_0^1 K_n(\nu,\alpha)\, A_1'(\alpha)\, d\alpha \quad . \tag{B.2}$$

The convergence properties of (B.2) are examined by investigating the series

$$H(\nu,\alpha;\lambda) = -\sum_{n=1}^{\infty} \lambda^{n-1} K_n(\nu,\alpha) \quad . \tag{B.3}$$

Define the norm of the kernel by

$$\| K^2 \| = \int_0^1 \int_0^1 K^2(\nu,\alpha)\, d\nu\, d\alpha$$

$$= \int_0^1 F^2(\nu)\, d\nu = \int_0^1 G^2(\alpha)\, d\alpha \le N^2 \quad , \tag{B.4}$$

where N^2 is an upper bound and

$$F^2(\nu) = \int_0^1 K^2(\nu,\alpha)\, d\alpha; \quad G^2(\alpha) = \int_0^1 K^2(\nu,\alpha)\, d\nu \quad . \tag{B.4a}$$

By Schwartz' inequality, it follows that

$$|K_{n+2}(\nu,\alpha)| \le |F(\nu)|\,|G(\alpha)|\, N^n \quad . \tag{B.5}$$

Inserting (B.5) into (B.3) we find

$$|H(\nu,\alpha;\lambda) - K(\nu,\alpha)| \leq |\lambda|\ |F(\nu)|\ |G(\alpha)| \sum_{m=0}^{\infty} |\lambda N|^m \quad . \tag{B.6}$$

The geometric series in (B.6) converges if

$$|\lambda N| < 1 \quad . \tag{B.7}$$

It remains now to estimate the upper bound N. Using the behavior of $X(-\nu)$ near $\nu = 0$ and $\nu = 1$ and of $g(c,\nu)$ from Ref. 4, it readily follows from (B.4) that

$$||K^2|| \leq \frac{e^{-2t}}{4(1-c)^2} = N^2 \quad . \tag{B.8}$$

Substitution of (B.1) and (B.8) in (B.7) yields

$$1 < \frac{4e^t}{c} \quad , \tag{B.9}$$

where t and c are related by the criticality condition and Eq. (9). Since in practical applications $c < 2$, the convergence condition of Eq. (B.9) is clearly satisfied.

2. Error Analysis:

An upper bound of the error introduced by keeping the first n terms of the series in (B.2) will now be estimated. First we note that $A_1'(\nu) \leq 0$ and the kernel $K(\nu,\alpha)$ is positive for $c > 1$. From this it follows that $\phi_n(\nu) \leq 0$. In view of the fact that λ is also negative, the series in (B.2) is alternating and can be written as

$$\sum_{n=1}^{\infty} \lambda^n \phi_n(\nu) = \sum_{n=1}^{\infty} (-1)^{n+1} a_n(\nu) \quad . \tag{B.10}$$

where $a_n(\nu)$ is positive and $a_{n+1} < a_n$. By the well-known theorem of convergent alternating series of this type, the error made in stopping at n terms is less in absolute value than the first term neglected. Thus, if we denote the absolute value of the error by $\epsilon_n(\nu)$, then

$$\begin{aligned} \epsilon_n(\nu) &< a_{n+1}(\nu) \\ &< |\lambda|^{n+1} |\phi_{n+1}(\nu)| \quad . \end{aligned} \tag{B.11}$$

Using the definition of $\phi_n(\nu)$ and Eq. (B.5) we find

$$| \phi_{n+1}(\nu) | \leq |F(\nu)||N|^{n-1} \int_0^1 |G(\alpha)||A_1'(\alpha)| \, d\alpha \quad . \tag{B.12}$$

A rough estimate of the upper bound of the factors on the right of (B.12) gives

$$| \phi_{n+1}(\nu)| \leq \frac{|\nu_0|^2}{2} e^{-t} E_2(t) | X^3(0)|| A_1'(0)| |N|^{n-1} p(\nu) \quad , \tag{B.13}$$

where we have defined

$$p(\nu) = \frac{(|\nu_0|^2 + \nu^2) X(-\nu) g(c,\nu)}{1+\nu} \tag{B.13a}$$

and $E_2(t)$ is the exponential integral tabulated in Ref. 4. Finally, substitution of (B.13) in (B.11) and use of the explicit expressions for $X(0)$, $|\lambda N|$, and $A_1'(0)$ gives

$$\epsilon_n(\nu) < 2a_{0+} \left(\frac{c}{4}\right)^{n+1} |B| \frac{e^{-nt} E_2(t)}{|\nu_0|^2} p(\nu) \quad , \tag{B.14}$$

where B is the constant given in Eq. (49a). In particular, the upper bound of the error in the first approximation is

$$\epsilon_0(\nu) < 2a_{0+} \frac{c}{4} \frac{|B|}{|\nu_0|^2} E_2(\nu) p(\nu) \tag{B.15}$$

and the expansion coefficients can be written as

$$A'(\nu) = A_1'(\nu) + \epsilon_0(\nu) \quad . \tag{B.16}$$

Furthermore the actual value of $A'(\nu)$ lies between $A_1'(\nu)$ and $A_1'(\nu) + \epsilon_0(\nu)$. In Table BI we give values for the lower and upper bounds of $A'(\nu)$, viz,

$$\underline{A'}(\nu) = A_1'(\nu)$$

$$\overline{A'}(\nu) = A_1'(\nu) + \frac{c}{4} \frac{|B|}{|\nu_0|^2} E_2(\nu) p(\nu) \quad ,$$

where we have used the normalization of Eq. (53).

Table BI

BOUNDS FOR $A'(\nu)$

ν	c = 2.00		c = 1.10	
	$\underline{A}'(\nu)$	$\overline{A}'(\nu)$	$\underline{A}'(\nu)$	$\overline{A}'(\nu)$
0	-0.586	-0.430	-0.160	-0.158
0.4	-0.143	-0.0922	-0.0814	-0.0806
0.8	-0.0303	-0.0108	-0.0368	-0.365
1.00	0.00	0.00	0.00	0.00

The exponential nature of Eq. (B.14) indicates that these bounds will be reduced considerably in the next higher approximations. Also, since the contribution of terms involving $A'(\nu)$ on the observables $\rho(x)$ and $\psi(x,\mu)$ was shown to be small for $1 \leq c < 2$, small errors in the coefficients can be neglected in the computations of these quantities.

Another result of interest follows from the above discussion. By taking the limit of Eq. (B.16) as $c \to 1$ and recalling that in this limit both t and $|\nu_0|$ become infinite, we find

$$\lim_{c \to 1} A'(\nu) = 0 \tag{B.17}$$

from which we conclude that $A(\nu) \equiv 0$ at $c = 1$.

3. Another Approximation Technique:

A method which gives higher-order estimates to the Fredholm Equation

$$A'(\nu) = A_1'(\nu) + \lambda \int_0^1 K(\nu,\alpha)\, A'(\alpha)\, d\alpha \tag{35}$$

consists of approximating the integral term by a Taylor expansion of $A'(\alpha)$ about $\alpha = \nu$:

$$A'(\alpha) = A'(\nu) + (\alpha - \nu)\frac{dA'(\nu)}{d\nu} + \cdots \quad . \tag{B.18}$$

Since the zeroth and first approximations to $\psi(x,\mu)$ correspond to $A'(\nu) = 0$ and $A'(\nu) = A_1'(\nu)$, respectively, they remain unaltered. For the next approximation we keep the first term of (B.18). This gives

$$A'(\nu) = A_1'(\nu) + \lambda A'(\nu) \int_0^1 K(\nu,\alpha)\, d\alpha$$

or

$$A'(\nu) = \frac{A_1'(\nu)}{1 - \lambda \int_0^1 K(\nu,\alpha)\, d\alpha} \tag{B.19}$$

Keeping the first two terms of (B.18) results in a first-order differential equation for $A'(\nu)$:

$$\frac{dA'(\nu)}{d\nu} + P(\nu)A'(\nu) = Q(\nu) \quad , \tag{B.20}$$

where

$$P(\nu) = \frac{\lambda \int_0^1 K(\nu,\alpha)\, d\alpha - 1}{\lambda \int_0^1 (\nu-\alpha)\, K(\nu,\alpha)\, d\alpha} \tag{B.20a}$$

and

$$Q(\nu) = - \frac{A_1'(\nu)}{\lambda \int_0^1 (\nu-\alpha)\, K(\nu,\alpha)\, d\alpha} \quad , \tag{B.20b}$$

which are known functions of ν. Since

$$A_1'(1) = 0 \quad \text{and} \quad K(1,\alpha) = 0$$

it follows that

$$A'(1) = 0 \quad . \tag{B.21}$$

The solution of (B.20) subject to (B.21) is

$$A'(\nu) = \int_1^{\nu} Q(\nu') \exp\left(- \int_{\nu'}^{\nu} P(s)\, ds \right) d\nu' \quad . \tag{B.22}$$

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