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COMBINED FORCED AND FREE TURBULENT CONVECTION IN A VERTICAL TUBE

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ANL-6528
Engineering and
Equipment
(TID-4500, 17th Ed.)
AEC Research and
Development Report

ARGONNE NATIONAL LABORATORY
9700 South Cass Avenue
Argonne, Illinois

COMBINED FORCED AND FREE TURBULENT CONVECTION IN A VERTICAL TUBE

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January 1962

Technical Report No. 12
Atomic Energy Commission Contract AT(11-1)-1026
Argonne National Laboratory Subcontract 31-109-38-704

Operated by The University of Chicago
under
Contract W-31-109-eng-38

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ABSTRACT

An analytical study was made of turbulent heat transfer in a vertical circular tube under the conditions of combined forced and free convection with uniform heat flux at the wall.

The basic conservation laws are reduced to three coupled, linear integro-differential equations in which the parameters are: Prandtl number Pr; Rayleigh number Ra; and friction Reynolds number Re^*. The following values of these parameters were investigated: Pr = 1,10,100; Ra = 0,16,81, 100,256,400,625,800; and $Re^* = 0, 10^3, 10^4, 10^5$.

The IBM-704 digital computer was used to solve 3 equations for the fully developed velocity profile, temperature profile, and pressure drop when the above parameters were fixed in value. An extension of these results yields the mixed-mean-to-wall temperature difference, Nusselt number, and Reynolds number for a given problem.

The case of pure forced-convection, laminar heat transfer is realized when $Re^* = 0$ and Ra = 0. The solution for this case checked exactly with well-known previous results.

Ordinarily, $Re^* = 0$ implies that the wall shear stress T_w is zero. This is not so in this problem, for Re^* has no meaning in laminar flow. It is made zero in order to have a zero eddy diffusivity, i.e., setting $Re^* = 0$ in laminar-flow problems is just a convenience in the numerical solution, and should not be used for any further implications.

Problems of laminar heat transfer under combined forced and free convection are realized by making $Re^* = 0$. Here again the solution checked previous results exactly.

For pure forced-convection, turbulent heat transfer problems, Ra = 0. On comparing the results upon solution of this type of problem with the universal velocity and temperature profiles, it was found that they fell below the accepted curves in the buffer zone and turbulent core, although agreement was obtained in the laminar sublayer. This discrepancy is due to the fact that too high a value of eddy viscosity is used in the buffer zone and in part of the turbulent core.

Too high a value of ϵ_M/ν also increases the Nusselt number above that calculated by the Dittus-Boelter equation. However, these results for the Nusselt number are fairly close to the upper limit of experimental values.

The results for turbulent heat transfer in combined forced- and free-convection problems showed that the pressure drop parameter C and the Nusselt number Nu increase by approximately an order of magnitude as Re* increases from 0 to 10^3 to 10^4 to 10^5. The Prandtl number Pr had a smaller effect on C and Nu, such that the former decreased and the latter increased as Pr increased. Increasing Re* decreased the mixed-mean-to-wall temperature difference ϕ_M by approximately an order of magnitude for the Re* change given above. Increase of Pr had a smaller effect on decreasing ϕ_M than increasing Re*.

The temperature-difference profile tended to become smaller and flattened as Re* increased. Increase of the Prandtl number had the same effect, but to a lesser extent. The temperature difference at the center of the tube approached zero as the Rayleigh number increased.

The velocity in the center of the tube was lowered and became more and more negative as the Prandtl number decreased and as the Rayleigh number increased. Increase of Re* had the same effect, i.e., lowered and negative velocities in the center of the tube, for Pr = 1; but, for Pr = 10 or Pr = 100, the velocity in the center of the tube became less negative when Re* was increased from 10^4 to 10^5.

INTRODUCTION

Convection heat transfer is accomplished by virtue of the movement of a fluid, which carries heat to or from a surface. The convecting fluid may move solely because parts of it have a different density than other parts. For most fluids a lower density results when the temperature is increased. The hotter part will rise due to an increase in the buoyant force, and will be replaced by the colder and more dense parts. Convection which occurs due to this natural movement of a fluid is called "free convection." More generally, free convection is due to a difference of body forces in different parts of a fluid. These body forces may come about because of gravity or rotation in the most common applications of engineering problems.

If, in addition, the convecting fluid is pumped or blown past the heat-transmitting surface, a situation of combined "forced" and free convection exists. The term forced convection is used to describe the limiting situation when the buoyant force is negligible compared with the pumping force.

There are many other ways in which a convection heat transfer problem may be classified. Conditions may be steady or unsteady. The flow may be either laminar or turbulent. The heat-transmitting surface may be at a constant temperature, may have a specified temperature variation along its length, or may have a uniform or specified heat flux along its length. The flow may be fully enclosed or external to a surface. If the fluid is flowing in a tube, the cross-sectional shape may be round, rectangular, square, triangular, annular, or some other shape. At a cross section, velocity and temperature profiles may be similar to the corresponding profiles at other cross sections (known as the region of "fully developed" flow and heat transfer) or they may be changing. In addition, uniform or nonuniform volume heat sources may or may not be present within the fluid. Finally, the tube may be inclined at any angle with the vertical.

The problem considered in the present study is one of combined forced and free convection in turbulent flow. The configuration chosen was that of a vertical, round tube with upward flow and uniform heat flux at the wall. Volume heat sources will not be considered. Only the case of fully developed flow and heat transfer was treated analytically and compared with available results.

This problem has practical importance in the cooling of nuclear reactors and turbine blading, and in process heat transfer. Its solution will show the effect of free convection on turbulent forced convection; and also the effect of turbulence in laminar flow and heat transfer, combined forced- and free-convection problems. The solutions to this fairly general problem can also be used to check the following special cases:

1. pure forced convection, laminar flow;
2. combined forced and free convection, laminar flow;
3. pure forced convection, turbulent flow.

SURVEY OF THE LITERATURE

Heat transfer applications in which both forced and free convection are present, i.e., in channels in which the flow is parallel to the direction of the gravity or centrifugal body force, have been rather recent. As a result, the characteristics of such systems have been determined, primarily during the last 20 years, and much interest in this field continues at the present time.

Theoretical Investigations

All of the theoretical investigations have been for the laminar flow case. Martinelli and Boelter(30)* presented an analysis for developing flow. They concluded that the flow conditions immediately adjacent to the solid-fluid interface apparently control the rate of heat transfer. Pigford(41) examined the same case, but included the influence of temperature on viscosity as well as on density. The Nusselt numbers calculated increased substantially above that for the case of constant viscosity. The results were about the same as those predicted earlier by Martinelli and Boelter when Pigford considered the viscosity to remain constant.

Ostroumov(38) treated the round-pipe problem for the cases in which the axial temperature gradient was constant, both positive and negative. Many other solutions were given for different geometries, but internal heat sources were not considered nor were Nusselt numbers calculated. Hallman(18) agreed with Ostroumov's solution of the case for constant wall heat flux addition with upward flow, but pointed out an error in the calculation of the mixed-mean-to-wall temperature difference because of an incorrect integration. In addition to predicting velocity and temperature profiles, Hallman also calculated Nusselt numbers and pressure drops. He treated cases in which volume heat sources were present as well as those in which volume heat sources were not present.

Ostrach's analysis(37) was for 2 plane, parallel surfaces in the direction of the body force with linearly varying wall temperature. Viscous dissipation was included in the energy equation. It was found that a modified Rayleigh number (usual Rayleigh number multiplied by the reciprocal of the specific heat ratio) and a dimensionless parameter (usual Rayleigh number multiplied by $\beta f_x b/C_p J$)** were of significance in this problem. Representative dimensionless velocity and temperature distribution were given, from which Nusselt numbers were calculated.

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*Numbers in parenthesis refer to the BIBLIOGRAPHY.

**The nomenclature is given in Appendix A.

Various other geometries and cases may be summarized briefly in the following paragraph.

Han[19] considered the case of rectangular tubes with uniform axial temperature gradient and uniform peripheral wall temperature by applying the method of undetermined coefficients and using a double Fourier series. Tao[54] introduced a complex function, directly related to the velocity and temperature fields, in treating the case of a vertical channel of constant axial wall temperature gradient, with or without heat generation. The analysis of Lu[27] was for heat-generating flow, i.e., with the presence of volume heat sources, in vertical pipes with circular sector cross sections. He applied the finite Fourier sine transform and finite Hankel transforms to the nondimensionalized Navier-Stokes and energy equations. Sparrow, Eichhorn and Gregg[53] presented an analysis for boundary layer flow, employing similar solution theory, and Acrivos[1] utilized the Polhausen-Von Karman integral method in considering external flows along a vertical isothermal plate. Morton's analysis[33] was for uniformly heated horizontal pipes with flow at low Rayleigh numbers. His approximate solution was based on expanding the dimensionless Stokes stream function, dimensionless velocity, and dimensionless temperature difference as power series in the Rayleigh number. He found that the Nusselt number is increased by the secondary flow resulting from buoyancy, and the increase depends upon the Rayleigh-Reynolds product squared.

Experimental Investigations

Not quite so much effort has been expended in obtaining experimental data in combined forced and free convection as in the special cases of pure forced or free convection. These may be considered limits at either end of the case of combined forced and free convection. McAdams'[31] and Jakob's[25] chapters on these subjects provide a wealth of references for these limiting cases.

Hallman[18] designed and built and experimental apparatus to measure the laminar heat transfer in a vertical tube with uniform wall heat flux and no internal heat generation. The experimental, fully developed Nusselt numbers checked very well with his analysis for positive Rayleigh numbers in the range from 27 to 2700. The Nusselt numbers fell about 10% above his predicted curve for negative Rayleigh numbers ranging from -50 to -115. Thermal entrance lengths for positive Rayleigh numbers were found to be shorter than for pure forced convection, whereas those for negative Rayleigh numbers were found to be longer than those for pure forced convection. Hallman also observed a transition to a fluctuating flow under certain conditions. Asymmetric distributions of wall temperature were found for negative Rayleigh numbers, which became more severe as the Rayleigh number became more negative.

Clark and Rohsenow[5] found that the effects of free convection on the nonboiling heat transfer process were significant; these effects caused a transition from laminar to turbulent flow at the surface to occur at Reynolds numbers in the range from 60,000 to 100,000. Local surface coefficients of heat transfer were presented for degassed, distilled water flowing upward in a vertical tube, 9.4 in. long and 0.180 in. in inside diameter.

Wetjen[61] reported some measurements of average heat transfer coefficients in water and glycerine flowing in an upward direction through a vertical tube with a length-to-diameter ratio of 50. No quantitative evaluation is possible because the published data are insufficient for the calculation of the parameters used in the present investigation.

Eckert, Diaguila, and Curren[11] and Eckert, Diaguila, and Livingood[12] conducted experiments in turbulent, combined forced- and free-convection flow of air with length-to-diameter ratios ranging from 5 to 40. Limits of the forced-flow regime, free-flow regime, and mixed-flow regime were established and were found to depend on the Reynolds and Grashof numbers. In addition to their own data, they used those of Martinelli and Boelter,[30] and of Watzinger and Johnson.[59]

Superposed forced and free convection of air in stationary horizontal and vertical rectangular ducts was experimentally investigated by Altman and Staub.[2] Heat transfer and pressure-drop results were plotted as Colburn "j" factors and friction factors versus Reynolds numbers. When their data were compared with the criterion of Eckert, Diaguila, and Livingood,[12] the majority of the data fell into the mixed-flow region. The maximum values of the experimental heat transfer coefficient in this mixed-flow region were as much as twice the values calculated using free- or forced-convective relations alone.

Gross[15] and Gross and Van Ness[16] presented data for laminar heat transfer in a vertical tube in which some effects of combined forced and free convection were present. The data of Poppendiek[42] and Poppendiek and Winn[43] also showed some effects of free convection on forced convection.

Jackson, Harrison, and Boteler[24] performed experiments of superposed free and forced laminar convection for air in a vertical tube. Since the Martinelli and Boelter[30] equation did not correlate their data satisfactorily, they analyzed the system from an overall viewpoint. This analysis led to the derivation of an equation that fit the experimental data.

Holman and Boggs[23] recently conducted heat transfer experiments using Freon 12, near the critical state, in a vertical tube which was part of a natural-circulation loop. Their analysis, based on overall conditions, was for laminar and for turbulent flows. Experimental results were obtained for the case of turbulent flow only.

Collis and Williams[6] determined the importance of combined convection for heat transfer from horizontal wires for flow at very low Rayleigh numbers. They found that free convection is significant when, roughly speaking, the Reynolds number is less than the cube root of the Rayleigh number. Yuge[62] investigated heat transfer between spheres and air. He presented empirical formulas for forced, natural, and combined convection, and used a graphical procedure to predict the heat transfer in combined natural and forced convection.

In studying natural-convection instabilities, Hanratty, Rosen, and Kabel[20] injected a thin stream of dye into water flowing through a vertical tube. The flow field was visually observed, and the value of the Grashof number divided by the Reynolds number was used as a criterion to describe the velocity profile. Mori's experiments[32] showed, for forced convection on a horizontal isothermal plate, the effects of natural convection upon the local coefficient of heat transfer are less than 10% if $|Gr_x| \leq 0.083(Re_x)^{2.5}$. This estimate applies to both sides of the plate.

Van Putte[56] investigated heat transfer to water in the near-critical region by means of a natural-convection loop. The Dittus-Boelter equation was found to be inadequate for correlating his results.

A few investigators have conducted experiments for no net through flow and no internal heat generation in vertical pipes in which the hotter fluid is below the colder fluid. They are: Ostroumov;[38] Martin;[29] Eckert, Diaguila, and Curren;[11] Hahnemann;[17] Hartnett and Welsh;[21] and Slavnov.[51]

ANALYSIS

The problem to be analyzed is combined forced and free turbulent convection in a circular tube whose axis is parallel to the direction of the body force. There is to be a net through flow.

Assumptions

In addition to the description of the problem given above, the following assumptions are made:

1. Axial symmetry exists for the momentum and heat transfer.
2. All fluid properties, except density, are constant in the expression for body force. A mean density is used for all other density terms.
3. Viscous dissipation and axial heat conduction are negligible compared with the heat conduction in the radial direction.
4. There is a uniform heat flux at the wall.
5. There are no volume heat sources.
6. The velocity and temperature profiles are fully developed.
7. There is single-phase flow.
8. The eddy diffusivities of momentum and heat are in constant proportion.
9. The eddy diffusivity of momentum is given by a modification of Reichardt's equation[45,46] and is based on the experimental data of Nikuradse,[35] Reichardt,[45,46] Nunner,[36] Laufer,[26] and many others. (See Appendix B.)

Basic Equations

The basic equations employed are the continuity, Navier-Stokes, and energy equations in cylindrical coordinates. An equation of state is also used.

On the basis of the above assumptions and description of the problem, the conservation laws reduce to

$$\frac{\partial u}{\partial x} = 0 \quad , \text{Continuity} \tag{1}*$$

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*The nomenclature is given in Appendix A.

$$\frac{\partial p}{\partial x} + \rho\frac{g}{g_c} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\left[\frac{\mu}{g_c} + \frac{\rho_m \epsilon_M}{g_c}\right]\frac{\partial u}{\partial r}\right) \tag{2}$$

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \psi} = 0 \tag{3}$$

Navier-Stokes (Momentum)

$$\rho_m c_p u \frac{\partial t}{\partial x} = \frac{1}{r}\left(r\left[k + \rho_m c_p \epsilon_H\right]\frac{\partial t}{\partial r}\right) \quad \text{Energy.} \tag{4}$$

The equation of state to be used is (see Appendix C)

$$\rho = \rho_W\left[1 - \beta\,(t - t_W)\right] \quad . \tag{5}$$

Development of Equations

Equation (1) indicates that

$$u = u(r) \quad . \tag{6}$$

Further, equations (3) indicate that

$$p = p(x) \quad . \tag{7}$$

If these facts and equation (5) are utilized, equation (2) can be written as

$$\frac{1}{r}\frac{d}{dr}\left(r\left[\frac{\mu}{g_c} + \frac{\rho_m \epsilon_M}{g_c}\right]\frac{du}{dr}\right) + \rho_W\,\beta\frac{g}{g_c}\left(t - t_W\right) = \frac{dp}{dx} + \rho_W\frac{g}{g_c} \quad . \tag{8}$$

The boundary condition of uniform wall heat flux plus the assumptions of constant specific heat and a fully developed temperature profile require that

$$\frac{\partial t}{\partial x} = A \text{ (a constant)} \quad . \tag{9}$$

Equation (9) is developed in Appendix D, in which a new variable θ is introduced and defined as

$$\theta = \theta(r) \equiv t(x,r) - t\left(x,\frac{D}{2}\right) = t - t_W \quad . \tag{10}$$

In terms of this new variable, equations (4) and (8) become, respectively,

$$\rho_m c_p u A = \frac{1}{r}\left(r\left[k + \rho_m c_p \epsilon_H\right]\frac{d\theta}{dr}\right) \tag{11}$$

and

$$\frac{1}{r}\frac{d}{dr}\left(r\left[\frac{\mu}{g_c} + \frac{\rho m \epsilon M(r)}{dr}\right]\frac{du(r)}{dr} + \rho_W \beta \frac{g}{g_c}\theta(r) = \frac{dp(x)}{dx} + \rho_W \frac{g}{g_c} \quad (12)$$

The density ρ_W appearing in this last equation is actually a function of x. We will consider the solution of this equation together with equation (11) to obtain u and θ at a fixed value of x; thus, we shall consider ρ_W constant. By evaluating the solutions at different values of x and by using appropriate physical properties at each x, we obtain results consistent with our assumption of "locally fully developed" flow and heat transfer. With this procedure it is seen that the right-hand side of equation (12) is a function of x alone, and the left-hand side is a function of r alone; hence, each side may be set equal to some constant. Thus,

$$\frac{dp}{dx} + \rho_W \frac{g}{g_c} = -\frac{32\, u_m \mu C}{D^2\, g_c} \quad \text{(a constant)} \qquad (13)$$

and

$$\frac{1}{r}\frac{d}{dr}\left(r\left[\frac{\mu}{g_c} + \frac{\rho_m \epsilon_M}{g_c}\right]\frac{du}{dr}\right) + \rho_W \beta \frac{g}{g_c}\theta = -32\, u_m \mu C / D^2\, g_c \quad . \qquad (14)$$

The pressure-drop parameter C in these equations was chosen because it takes on the value of unity for the special case of pure forced-convection laminar flow, as described by Hallman.[18]

In order systematically to solve the general set of problems described, equations (11) and (14) are nondimensionalized by using the following defined terms:

$\eta \equiv 2r/D$	dimensionless radius	(15)
$\phi \equiv 2k\theta / \rho_W u_m c_p A D^2$	dimensionless temperature difference	(16)
$U \equiv u/u_m$	dimensionless velocity	(17)
$Ra \equiv \rho_m \rho_W \beta g c_p A D^4 / 16 \mu k$.	Rayleigh number (dimensionless)	(18)

Thus, equations (12) and (14) become

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\left[1 + \frac{\epsilon_H}{\alpha_m}\right]\frac{d\phi}{d\eta}\right) = \frac{U}{2} \qquad (19)$$

and

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\left[1+\frac{\epsilon_M}{\nu_m}\right]\frac{dU}{d\eta}\right) = -2Ra\phi - 8C \quad . \tag{20}$$

In accordance with assumption (8), we will use

$$\frac{\epsilon_H}{\epsilon_M} \equiv \sigma = \frac{6}{\pi^2}\sum_{\eta=1}^{\infty}\frac{1}{\eta^2 \exp\left(.01\,\eta^2/Pr\right)} \tag{21}$$

as given by Lykoudis.[28] Thus we may rewrite equation (19) as

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\left[1+\sigma Pr\,\frac{\epsilon_M}{\nu_m}\right]\frac{d\phi}{d\eta}\right) = \frac{U}{2} \tag{22}$$

Boundary Conditions

The following boundary conditions are used:

$$U(1) = 0 \tag{23}$$

$$\phi(1) = 0 \tag{24}$$

$$\frac{dU(0)}{d\eta} \equiv U'(0) = 0 \tag{25}$$

$$\frac{d\phi(0)}{d\eta} \equiv \phi'(0) = 0 \tag{26}$$

These conditions come from the physical problem. Equations (23) and (24) state that the fluid velocity and temperature at the wall of the tube ($\eta = 1$) are equal, respectively, to the wall velocity and wall temperature, i.e., an application of the no-slip boundary condition in a continuum. Equations (25) and (26) come from the fact that heat and momentum are not transferred across the center line ($\eta = 0$) of the tube, due to axial symmetry, resulting in a zero slope for the temperature and velocity profiles at the center line.

Discussion of Equations

Equations (20) and (22) are the momentum and energy equations, respectively. These are 2 second-order, linear, coupled differential equations which are to be solved for U and ϕ as functions of η. Boundary conditions (23), (24), (25), and (26) are to be satisfied. In addition, the value of the pressure-drop parameter C may also be obtained if we use the following form of the continuity equation:

$$U_m = \int_0^1 U\eta \, d\eta = 1 \quad . \tag{27}$$

Two obvious parameters in our equations are the Prandtl number Pr and the Rayleigh number Ra. These are usual dimensionless groups. It should be noticed, however, that Ra is defined in terms of the axial temperature gradient A and that Ra = Gr Pr when the Grashof number Gr is also defined in terms of A [see Ostrach(37)] as $Gr \equiv \rho_m \rho_W \beta g AD^4/16\mu^2$.

An empirical equation for ϵ_M/ν is given in Appendix B, i.e.,

$$\epsilon_M/\nu = 0.0667 \, Re^* \, (0.5 + \eta^2) \, (1 - \eta^2) \qquad \text{for } 0 \le \eta < \left(1 - \frac{10}{Re^*}\right) \tag{28a}$$

$$\epsilon_M/\nu = 0 \qquad \text{for } \left(1 - \frac{10}{Re^*}\right) \le \eta \le 1 \quad , \tag{28b}$$

where

$$Re^* \equiv Du^*/\nu \quad , \quad \text{friction Reynolds number} \tag{29}$$

and

$$u^* \equiv \sqrt{\tau_W g_c/\rho_W} \quad , \quad \text{friction velocity} \quad . \tag{30}$$

Thus a third parameter, the friction Reynolds number Re*, is introduced.

Extension of Results

The solution of equations (20), (22), and (27) will yield U and ϕ as functions of η for a given set of values of the parameters: Ra, Pr, and Re*. The pressure-drop parameter will also be obtained.

These results may be extended by calculating the following useful quantities: the axial pressure drop $(p - p_0)/(x - x_0)$; the dimensionless mixed-mean-to-wall temperature difference ϕ_m; the Nusselt number Nu; and the Reynolds number Re.

Hallman(18) gives equations for the first three of these quantities. They are:

$$\frac{p - p_0}{x - x_0} = \frac{32\mu u_m C_0}{D^2 g_c} - \rho_{W_0} \frac{g}{g_c}\left[1 - \frac{\beta A}{2}(x - x_0)\right] \quad , \tag{31}$$

where the subscript "0" refers to a reference axial position along the tube at which the temperature and velocity profiles are fully developed:

$$\phi_m \equiv 2\int_0^1 \phi U \eta \, d\eta \quad , \tag{32}$$

and

$$Nu \equiv hD/k = -2\phi'(1)/\phi_m \quad . \tag{33}$$

An expression for the Reynolds number is derived in Appendix E. The result is

$$Re \equiv \frac{Du_m}{\nu} = \frac{\pm(Re^*)^2}{8C + 2Ra\phi_m} \quad , \tag{34}$$

where the + sign is for upward flow and the - sign is for net downward flow.

Method of Solution

At first, it was decided to use the formal method of Frobenius by expressing U and ϕ as infinite power series in η, and trying to determine the coefficients of these series. The indicial equations had repeated roots; therefore, the solutions could be expressed as

$$\phi = (\ln\eta)\sum_0 a_n\eta^n + \sum_0 C_n\eta^{n+m} \tag{35}$$

and

$$U = (\ln\eta)\sum_0 b_n\eta^n + \sum_0 d_n\eta^{n+\ell} \tag{36}$$

[see Wayland[60]].

Conditions (25) and (26), which state that the derivatives of these functions are to be zero at $\eta = 0$, could not be satisfied because of the $\ln\eta$ term in the equations for ϕ' and U', i.e.,

$$\phi' = (\ln\eta)\sum_0 n\, a_n\eta^{n-1} + \frac{1}{\eta}\sum_0 a_n\eta^n + \sum_0 (n + m)C_n\eta^{n+m-1} \tag{37}$$

$$U' = (\ln\eta)\sum_0 nb_n\eta^{n-1} + \frac{1}{\eta}\sum_0 b_n\eta^n + \sum_0 (n+\ell)d_n\eta^{n+\ell-1} \quad . \tag{38}$$

In the method attempted next, the assumed variation of ϵ_M/ν with η was simplified so that it was constant in the central section of the tube, zero in the laminar sublayer, and linear between the two. The momentum and energy equations could then be combined into a form of Bessel's equation, as in Hallman's analysis,[18] and a solution obtained for the central section and the laminar sublayer in terms of Bessel functions. An infinite series solution was used for the section between the above two. All the boundary conditions were able to be satisfied, but a tremendous amount of algebra was required to set up 29 linear algebraic equations in 29 unknowns. Even after these 29 equations are solved by means of a digital computer, much work would have to be done to finally obtain the velocity and temperature profiles, C, ϕ_m, Nu, and Re - all for only one set of parameters.

A third method of solution involved a still further simplification of ϵ_M/ν so that it was constant in the area between the laminar sublayer and the central section of the tube. This led to just a little less work than was involved in the second method.

A few other methods of solution were explored, such as assuming a velocity profile for the laminar sublayer and possibly for the buffer zone, too, but these did not work out well because the Reynolds number had to be known beforehand. Actually, Re is a function of the other parameters and quantities in a given problem, as shown by equation (34).

Finally, since it appeared that the use of a digital computer would be most desirable, the following method of solution was devised to exploit the advantages of such a machine by decreasing tremendously the amount of hand calculations necessary, and thereby decreasing the possibility of errors in such a long, tedious problem as this.

As stated previously, the independent parameters are Ra, Re*, and Pr. The friction Reynolds number Re* determines the value of ϵ_M/ν as a function of η, and the Prandtl number Pr determines the value of σ. With these quantities fixed, the method of solution proceeds as follows:

1. $$U_1 = 2(1 - \eta^2) \tag{39}$$

is assumed as an initial guess. This equation satisfies the continuity equation (27), as well as boundary conditions (23) and (25). It is the limiting case of the parabolic profile in pure forced-convection laminar flow.

2. Equation (22) is integrated with the aid of condition (26) to obtain

$$\phi' = \frac{\frac{1}{2}\int_0^n U_1 \eta \, d\eta}{\eta\left[1 + \sigma \Pr \frac{\epsilon_M}{\nu}\right]} . \tag{40}$$

3. Equation (40) is integrated subject to boundary condition (24) to obtain

$$\phi = \int_1^n \phi' \, d\eta \tag{41}$$

as a function of η.

4. Equation (20) is integrated to obtain

$$U_2' = \frac{-2Ra \int_0^n \phi \eta \, d\eta}{\eta \left[1 + \frac{\epsilon_M}{\nu}\right]} - \frac{4\eta C}{\left[1 + \frac{\epsilon_M}{\nu}\right]} \quad , \tag{42}$$

where condition (25) has been utilized.

The parameter C must have a numerical value if the digital computer is to be used. As previously discussed, C has a value of unity for the limiting case of pure forced-convection laminar flow. A particular numerical value of C will have to be assumed in a given problem.

5. Equation (42) is integrated subject to boundary condition (23) to obtain

$$U_2 = \int_1^n U_2' \, d\eta \tag{43}$$

as a function of η.

6. The continuity equation (27) is checked to see if it is satisfied. If U_2 does not satisfy this equation, the value of C is to be changed until it does.

7. The function U_2 is checked against U_1 by comparing $U_2'(1)$ with $U_1'(1)$ to see if they are within 0.1% of each other.

8. Equations (32), (33) and (34) are used to calculate ϕ_m, Nu, and Re, respectively, if the above check is satisfactory. The results are then printed out as given in Step 10 below.

9. If the check in Step 7 is not satisfactory, U_2 is used as an initial assumption, replacing U_1 in going through the procedure again, starting at Step 2.

10. <u>Printout</u> Title

Problem Number

Parameters: Ra Re* Pr σ

η	U_2	U_1	ϕ
0	↓	↓	↓
↓			
1	0	0	0
C	ϕ_m	Nu	Re

It should be noted that both U_2 and U_1 are printed out to ensure that the criterion used for checking them in Step 7 is valid.

These steps may be shown in a simplified flow diagram Figure 1, for the calculation. A complete flow diagram is shown in Figure 2.

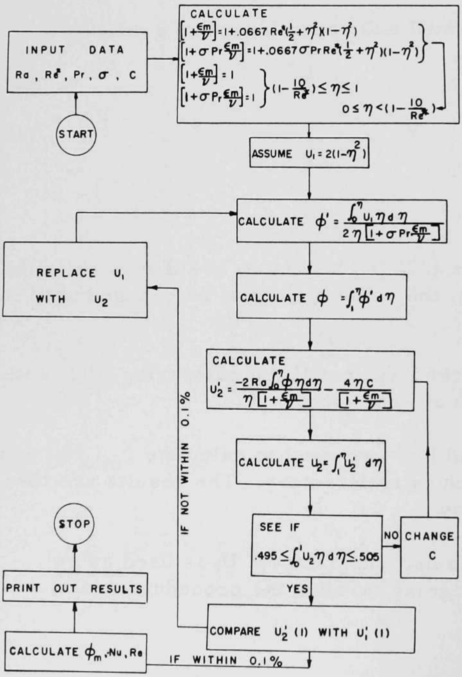

Figure 1

Simplified Flow Diagram of the Calculations

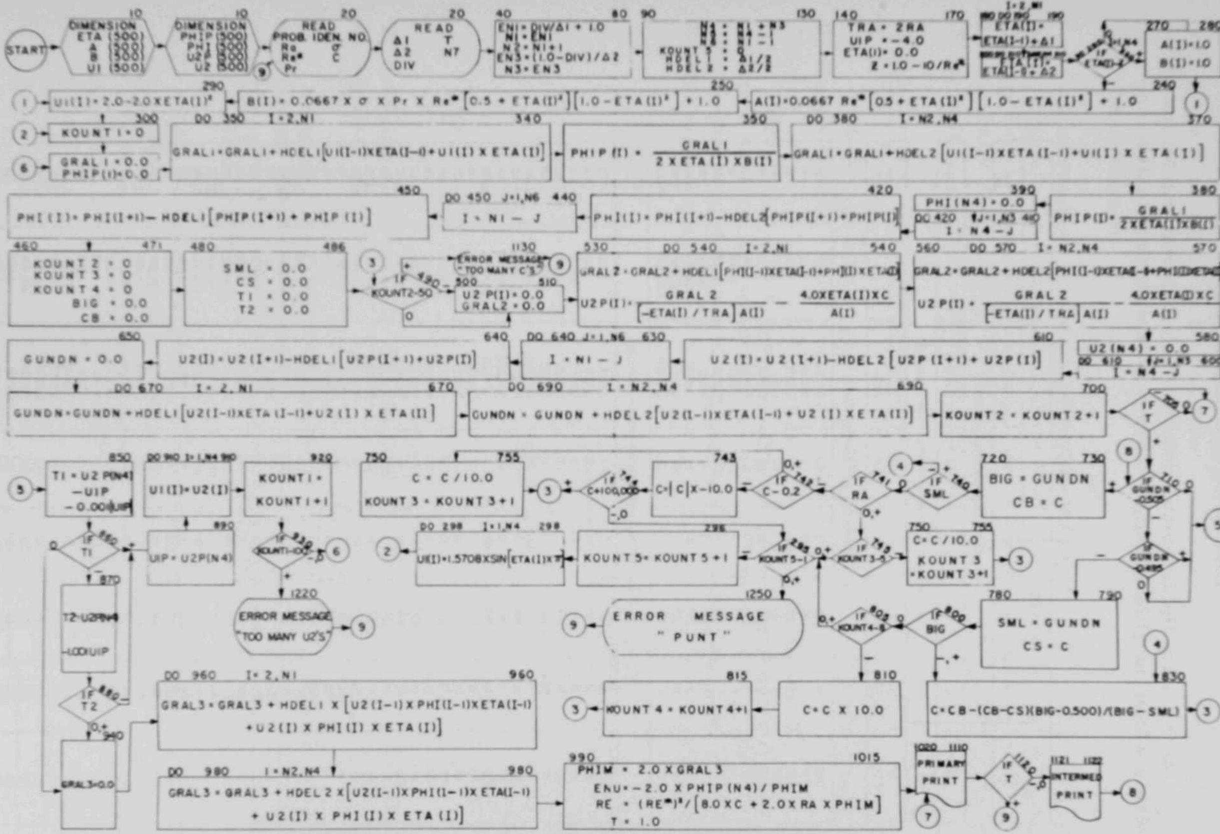

Figure 2. Flow Diagram of the Calculations

CALCULATED RESULTS

Table 1 lists the following results: C, $-\phi_m$, Re, and Nu for all problems which the IBM 704 digital computer was able to solve.

Table 1

RESULTS FOR ALL COMPLETED PROBLEMS

Problem No.	Input Parameters			Calculated Results			
	Ra	Re*	Pr	C	$-\phi_m$	Re	Nu
A. Laminar flow, pure forced convection							
2	0	0	-	1.0	0.1146	-	4.36
B. Laminar flow, combined forced and free convection							
52	81	0	-	3.13	0.0974	-	5.13
51	256	0	-	6.84	0.0761	-	6.57
C. Turbulent flow, pure forced convection							
119	0	300	1	4.1	0.019	2,760	26
120	0	300	10	4.1	0.0095	2,760	53
121	0	300	100	4.1	0.0083	2,760	60
122	0	500	1	6.1	0.013	5,100	39
123	0	500	10	6.1	0.0053	5,100	94
124	0	500	100	6.1	0.0045	5,100	111
47	0	10^3	1	10.1	0.0074	12,400	67
48	0	10^3	10	10.1	0.0030	12,400	165
3	0	10^4	1	69	0.0010	180,000	480
49	0	10^4	10	69	0.00032	180,000	1,560
42	0	10^5	1	560	0.00013	2,200,000	3,900
50	0	10^5	10	560	0.000025	2,200,000	20,300
D. Turbulent flow, combined forced and free convection							
83	16	10^3	1	17.1	0.0064	7,300	78
84	16	10^3	10	13.0	0.0030	9,600	168
95	16	10^3	100	12.5	0.0025	10,000	196
85	16	10^4	1	132	0.00090	94,000	550
86	16	10^4	10	90	0.00032	139,000	1,580
97	16	10^4	100	85	0.000025	148,000	2,000
87	16	10^5	1	1170	0.00011	1,100,000	4,440
88	16	10^5	10	690	0.000024	1,800,000	20,700
99	16	10^5	100	630	0.000014	2,000,000	36,500
43	81	10^3	1	35	0.0052	3,600	96
44	81	10^3	10	24	0.0028	5,200	177
101	81	10^3	100	22	0.0025	5,600	198
45	81	10^4	1	290	0.00084	43,000	600
46	81	10^4	10	167	0.00030	75,000	1,650
103	81	10^4	100	146	0.00024	85,000	2,000
41	81	10^5	1	2700	0.00012	470,000	4,300
40	81	10^5	10	1160	0.000023	1,100,000	21,800
105	81	10^5	100	910	0.000014	1,400,000	36,400
4	100	10^3	1	39	0.0053	3,200	95
5	100	10^3	10	27	0.0028	4,600	176
6	100	10^4	1	320	0.00087	39,000	570
7	100	10^4	10	190	0.00030	67,000	1,660
9	100	10^5	10	1290	0.000023	970,000	21,900
23	256	10^3	10	50	0.0028	2,500	176
107	256	10^3	100	48	0.0025	2,600	200
25	256	10^4	10	340	0.00031	36,000	1,600
109	256	10^4	100	310	0.00024	40,000	2,050
27	256	10^5	10	2200	0.000023	560,000	22,000
111	256	10^5	100	1600	0.000013	760,000	37,000
125	400	10^3	10	67	0.0031	1,870	162
126	400	10^3	100	68	0.0025	1,850	200
127	400	10^4	10	460	0.00035	27,000	1,440
128	400	10^4	100	440	0.00025	28,000	2,000
129	400	10^5	10	3000	0.000025	420,000	20,400
130	400	10^5	100	2200	0.000013	560,000	37,000
29	625	10^3	10	89	0.0036	1,400	138
113	625	10^3	100	99	0.0026	1,270	190
31	625	10^4	10	610	0.00042	20,500	1,200
115	625	10^4	100	640	0.00026	19,600	1,900
33	625	10^5	10	3800	0.000028	325,000	17,700
117	625	10^5	100	3100	0.000014	400,000	36,000
131	800	10^3	10	104	0.0040	1,210	124
132	800	10^3	100	122	0.0027	1,030	183
133	800	10^4	10	710	0.00047	17,600	1,060
134	800	10^4	100	790	0.00027	15,800	1,830
135	800	10^5	10	4400	0.000031	280,000	15,900
136	800	10^5	100	3800	0.000014	330,000	35,000

Part A consists of the problem in laminar flow, pure forced convection; part B has 2 problems in laminar flow, combined forced and free convection; part C consists of 12 problems in turbulent flow, pure forced convection; and, finally, part D has 47 problems in turbulent flow, combined forced and free convection.

The original IBM printout sheets for all these problems are kept with a copy of this thesis in a looseleaf binder at the School of Mechanical Engineering Library, Purdue University, Lafayette, Indiana. A Fortran listing, Table for Location of Variables, and a Share Assembly Program (SAP) Listing are also included in the binder.

Discussion and Comparison of Results

The 3 problems of parts A and B, Table 1, check exactly with Hallman's results[18] for laminar convection.

Problem 3 was chosen as typical of the problems in part C. The universal velocity and temperature profiles of Eckert and Drake[13] were used to check the values of U and ϕ in this problem. The equations needed for this comparison are

$$u^+ = \frac{Re}{Re*} (U) \quad , \tag{44}$$

$$(\Delta t^+) = 2 \; Pr \; Re* \; (\phi) \quad , \tag{45}$$

and

$$Y^+ = \frac{1}{2} Re* \; (1 - \eta) \quad . \tag{46}$$

These equations are easily derived from the definitions of the various dimensionless quantities.

Figure 3 shows the comparison of u^+ and $-(\Delta t^+)$ with the universal velocity and temperature profiles. The values in problem 3 fall below the universal profile in the turbulent core and buffer zone, but coincide exactly in the laminar sublayer.

Upon investigation of this discrepancy, it was found that the value of ϵ_M/ν used in the buffer zone and in part of the turbulent core was too high. The value $\epsilon_M/\nu = 0$ in the laminar sublayer is quite good, however. The variation of ϵ_M/ν with Y^+ is shown in Figure 4.

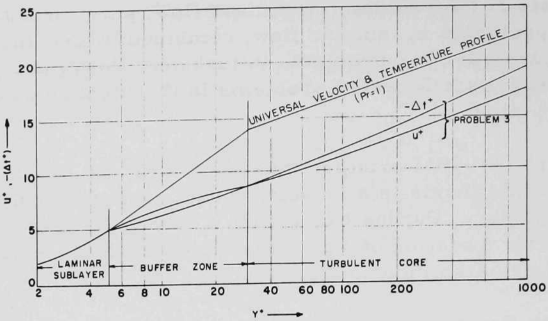

Figure 3. Comparison of Problem 3 Results with Universal Velocity and Temperature Profile

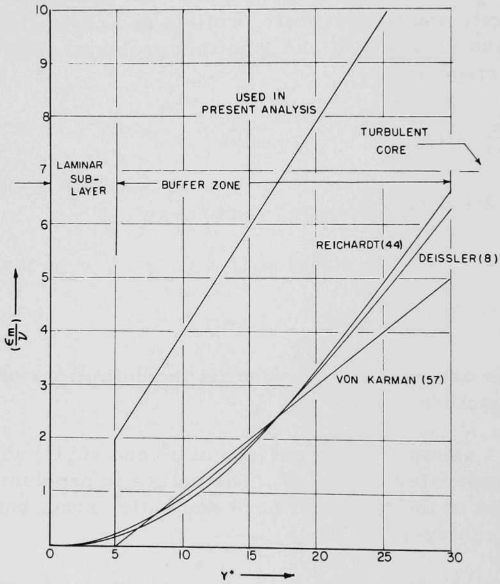

Figure 4. Variation of $\left(\frac{\epsilon_M}{\nu}\right)$ with Y^+.

All of the problems in part C with a Prandtl number of 1 or 10 are plotted in Figure 5, which shows the variation of Nusselt number with

Reynolds number. The Dittus-Boelter equation for pure forced-convection turbulent heat transfer is also shown, with dashed lines indicating the range of experimental results according to Giedt.[14] In both Prandtl-number comparisons, the results of the present analysis lie above the curves of the Dittus-Boelter equation. This discrepancy is due, again, to using too high a value of ϵ_M/ν in the buffer zone and part of the turbulent core.

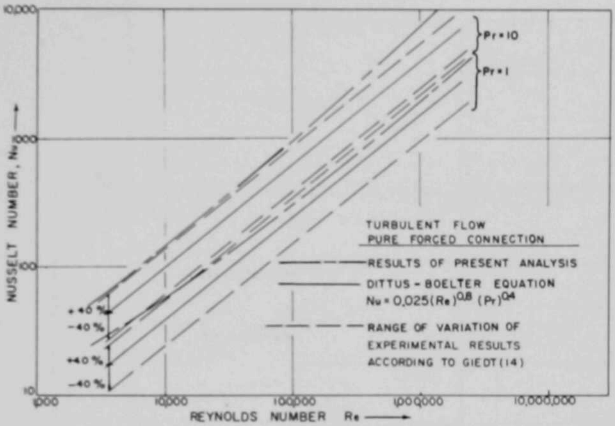

Figure 5. Comparison of Results with Dittus-Boelter Equation and Experiment

Figure 6 shows the effect of free convection on turbulent forced convection.

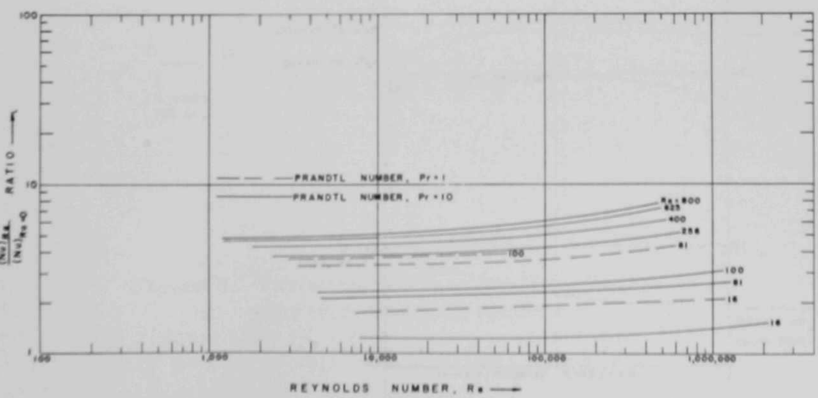

Figure 6. Variation of $\frac{(Nu)_{Ra}}{(Nu)_{Ra=0}}$ with Reynolds Number

These curves show that, for a given Rayleigh number, the ratio $(Nu)_{Ra}/(Nu)_{Ra=0}$ increases as the Reynolds number is increased and as the Prandtl number is decreased. At a given Reynolds number and Prandtl number, a higher ratio is obtained with a higher Rayleigh number.

Figures 7, 8 and 9 show results for the problems in turbulent flow, combined forced and free convection.

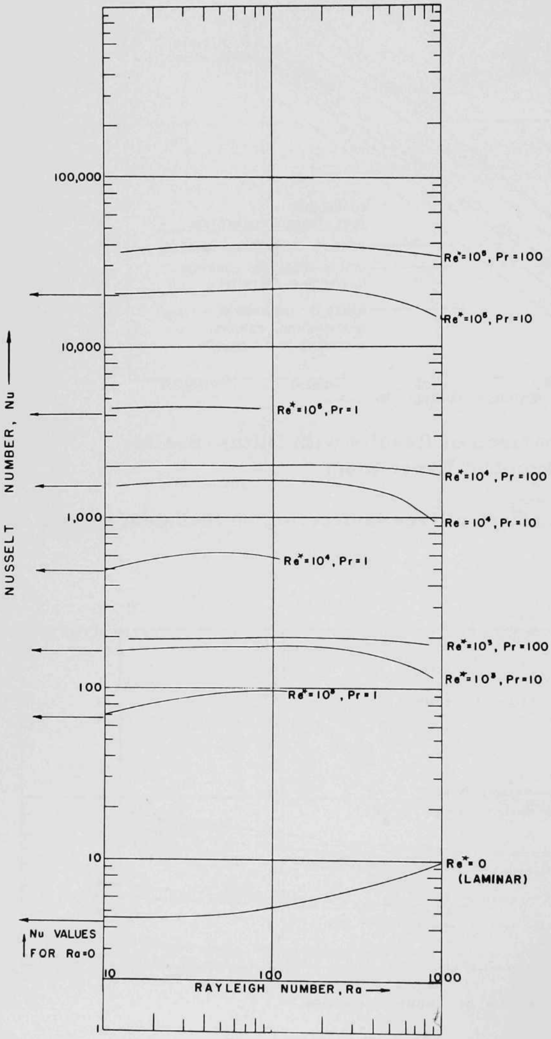

Figure 7

Variation of Nusselt Number with Rayleigh Number

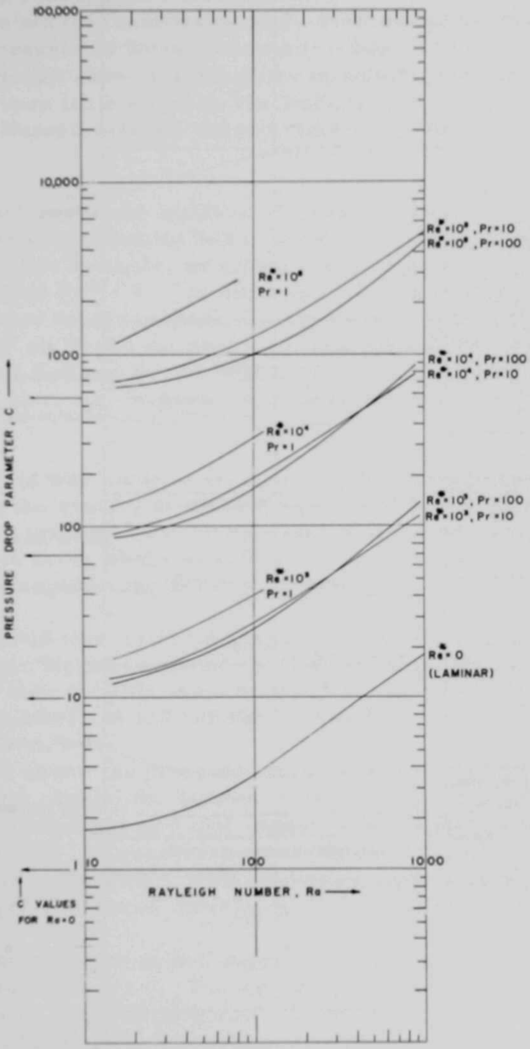

Figure 8. Variation of Pressure Drop Parameter with Rayleigh Number

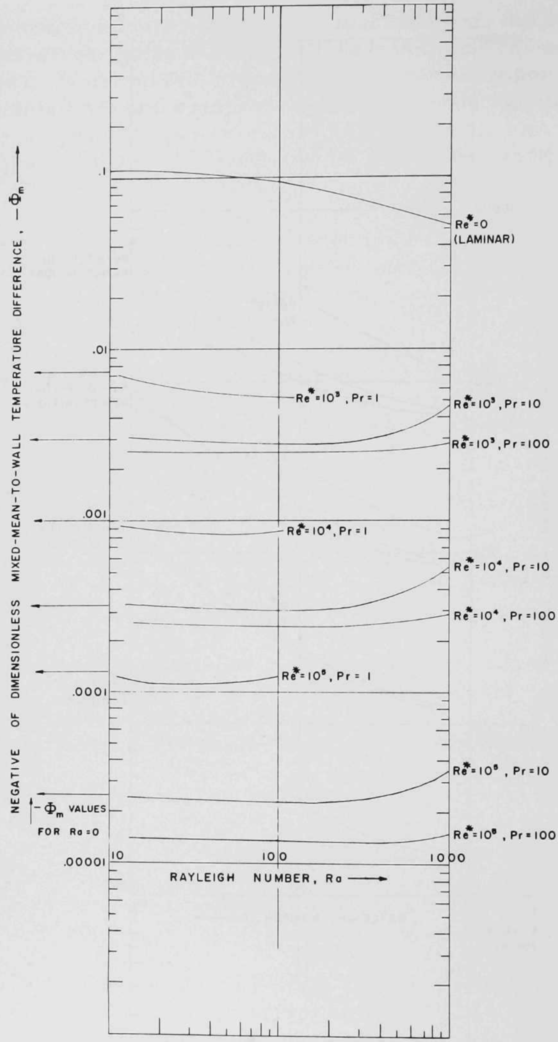

Figure 9. Variation of Mean Temperature Difference with Rayleigh Number

The Nusselt number is plotted against Rayleigh number in Figure 7. The lowest curve is for laminar flow. The Nusselt numbers are increased by approximately an order of magnitude as Re* is increased from 0 to 10^3

to 10^4 to 10^5. Increasing the Prandtl number has a smaller effect on increasing the Nusselt number than increasing Re*. Increasing the Rayleigh number continually increases the Nusselt number in laminar flow. The results for turbulent flow do not show this behavior; increasing the Rayleigh number seems to have very little effect on the Nusselt, and, indeed some decrease is noted in the Nusselt number above a Rayleigh number of approximately 400.

It was noticed in the solution of these problems with the IBM 704 digital computer that difficulty was experienced when the Rayleigh number was above 400. For instance, no solution was obtained for a problem in which Ra = 625 and Re* = 0. The iteration process seemed to oscillate. Similar action was noted for problems where Ra = 4096, 10^4, or 10^6 with any value of Re* or Pr; or for problems where Ra = 256, 400, 625, or 800 with any value of Re* and Pr = 1. Solutions were obtained for lower Rayleigh numbers, however, or when the Prandtl number is increased to 10 or 100.

An attempt was made to analyze what was happening within the IBM 704 during the running of these unsuccessful problems by examining the intermediate printout between successive iterations. It was found that the program was doing what was called for, but that the velocity profile oscillated or diverged in the iteration process.

It is evident that further work is required to explain this anomaly. Possibly, another digital computer, such as the IBM 7090, can handle this problem better than the IBM 704. At any rate, the results for problems with Rayleigh numbers of 400 and above should be suspect.

Figure 8 shows the pressure-drop parameter C plotted against Rayleigh number. Again, the bottom curve is for laminar flow, and C increases by approximately an order of magnitude as Re* is increased from 0 to 10^3 to 10^4 to 10^5. In general, decreasing the Prandtl number has a smaller effect in increasing C than increasing Re*. Increasing the Rayleigh number has a great effect on increasing C for all values of Re*.

The dimensionless mixed-mean-to-wall temperature difference ϕ_m is presented in Figure 9. The top curve is for laminar flow, and ϕ_m is decreased by an order of magnitude, approximately, as Re* is increased from 0 to 10^3 to 10^4 to 10^5. Increasing the Prandtl number has a lesser effect on decreasing ϕ_m than increasing Re*. Finally, increasing the Rayleigh number always decreases ϕ_m in laminar flow, but seems not to have much effect in turbulent flow. Again, the results for a Rayleigh number above 400 should be suspect.

Representative velocity and temperature profiles are shown in Figures 10 through 17.

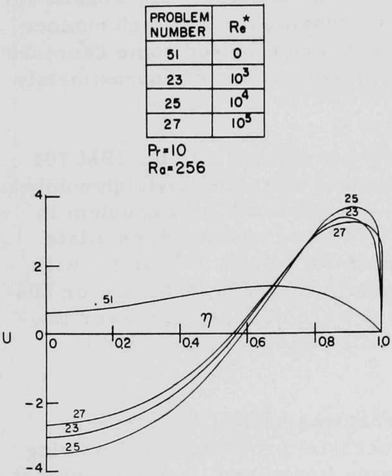

Figure 10. Velocity Profiles

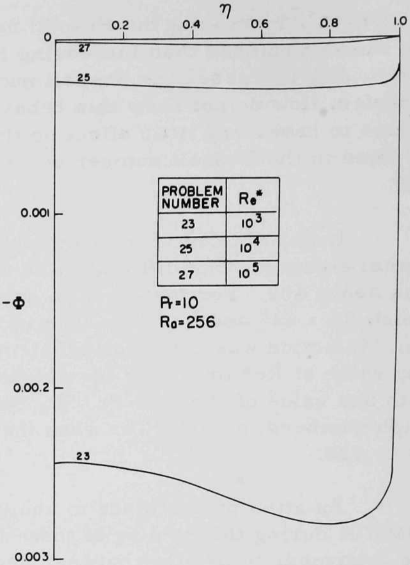

Figure 11. Temperature Difference Profiles

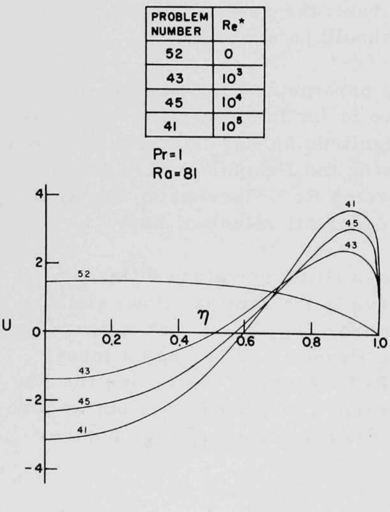

Figure 12. Velocity Profiles

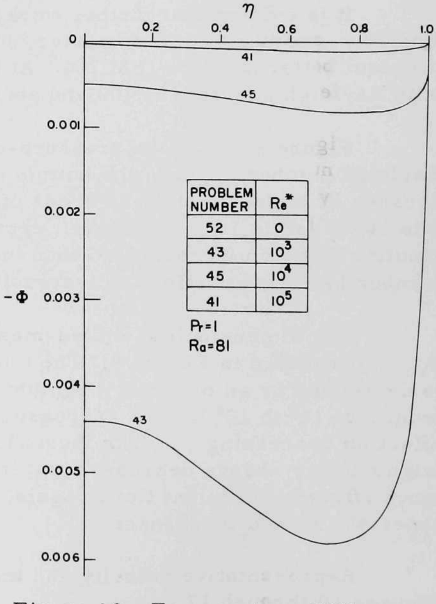

Figure 13. Temperature Difference Profiles

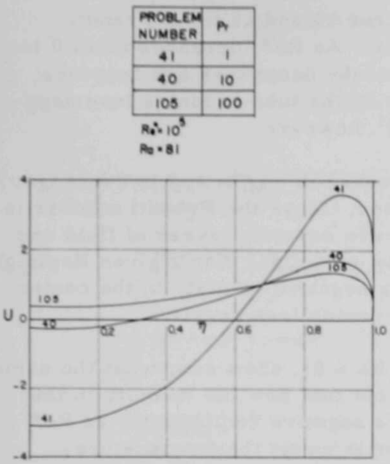

Figure 14. Velocity Profiles

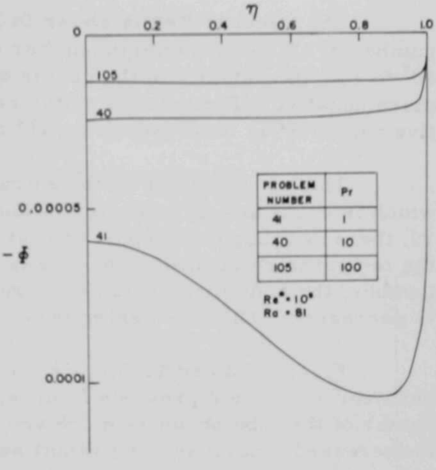

Figure 15. Temperature Difference Profiles

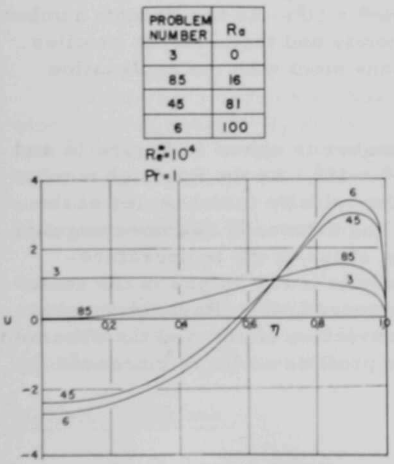

Figure 16. Velocity Profiles

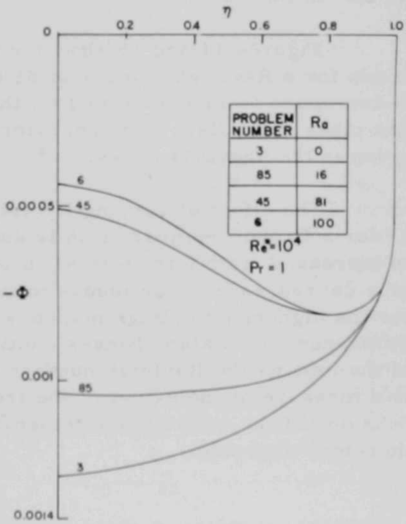

Figure 17. Temperature Difference Profiles

The effect of Re* is shown in Figures 10 and 11 for a Prandtl number of 10 and a Rayleigh number of 256. As Re* increases from 0 to 10^3 to 10^4, the velocity in the center of the tube decreases and becomes more negative. The velocity in the center of the tube becomes less negative when Re* is increased from 10^4 to 10^5, however,

This action is due to the extra increase in ϵ_M/ν and in $\sigma A(\epsilon_M/\nu)$, which is equal to ϵ_H/α, as Re* is increased. Since the Prandtl number is 10, there is a large transport of heat between adjacent layers of fluid and the temperature difference profile is made quite flat. For a given Rayleigh number, the body force tending to create a negative velocity in the center is decreased. Thus the center velocity becomes less negative.

Figures 12 and 13, for Pr = 1 and Ra = 81, show somewhat the same behavior as in the 2 previous figures, except that now the velocity in the center of the tube becomes more and more negative continuously as Re* is increased. Because the Prandtl number is unity, the temperature-difference profile is made not quite so flat as in the case for a Prandtl number of 10. The body force, therefore, is greater in the center of the tube, and the velocity consequently becomes more negative continuously as Re* is increased.

Figures 14 and 15 show the effect of using different Prandtl number fluids for a Rayleigh number of 81 and Re* = 10^5. As the Prandtl number is increased from 1 to 10 to 100, the velocity and temperature profiles are made more flat. This behavior is consistent with the explanation given in the previous cases.

The effect of varying Rayleigh number is shown in Figure 16 and 17 for a Prandtl number of unity and Re* = 10^4. As the Rayleigh number is increased from 0 to 16 to 81 to 100, the velocity in the center of the tube decreases in a continuous manner, and eventually becomes negative for the highest 2 Rayleigh numbers. The shape of the temperature-difference curve also changes continuously to lower values in the center of the tube as the Rayleigh number is increased. The Rayleigh number is a measure of the extent of the free-convective effect, and the observed behavior of the velocity and temperature profiles as Ra is increased is therefore expected.

SUMMARY AND CONCLUSIONS

The effect of the parameters Re*, Pr, and Ra on Nu, C, $-\phi_m$, U, and ϕ profiles is summarized in Table 2.

Table 2

SUMMARY OF EFFECTS IN COMBINED FORCED- AND FREE-CONVECTION PROBLEMS

Effect on → of ↓	Nu	C	$-\phi_M$	U Profile	ϕ Profile
Increasing Re* a) Pr 1	Increases	Increases	Decreases	Lowers velocity in center of tube making it more and more negative	Flattens curve and makes temperature difference approach zero
b) Pr 10 or 100	Increases	Increases	Decreases	Same as above except center velocity becomes less negative when Re* is increased from 10^4 to 10^5	Same as above
Increasing Pr a) Turbulent flow	Increases	Decreases	Decreases	Flattens curve	Flattens curve
b) Laminar flow	No effect	No effect	No effect	No effect	No effect
Increasing Ra a) Turbulent flow	Increases slowly for Ra up to 256	Increases	Decreases slowly for Ra up to 256	Lowers the velocity in the center of the tube, eventually making it negative	Makes the temperature difference in the center approach zero
b) Laminar flow	Increases	Increases	Decreases	Same as above	Same as above

All cases of laminar heat transfer, pure forced convection or combined forced and free convection, check exactly with accepted results. There is a discrepancy between accepted results and the cases of pure forced-convection turbulent heat transfer. It was therefore concluded that the values of eddy viscosity used were somewhat in error. Further investigation substantiated this conclusion, as shown in Figure 4.

It is a well-known fact that higher-Prandtl-number fluids will increase the Nusselt number in pure forced-convection turbulent heat transfer. This same result is obtained, with lower pressure drops, in combined forced- and free-convection turbulent heat transfer.

Recommendations

The following recommendations are made in order to extend the results of this study and to make them more accurate:

1. Isothermal wall conditions, negative Rayleigh numbers, and volume heat sources should be considered.

2. A more complete investigation should be made of the fact that the IBM 704 cannot handle problems with a high Rayleigh number and low Prandtl number, in general. Problems in which $Re^* = 10^6$, also, cannot be solved satisfactorily. Possibly, the program for the digital computation could be rewritten to overcome these difficulties.

3. A better equation for ϵ_M/ν should be obtained so that the universal velocity and temperature profiles may be more closely approached in pure forced-convection turbulent heat transfer problems.

4. The relation used for $\sigma = \epsilon_H/\epsilon_M$ is admittedly not very good, since it was developed for liquid metals, which have low Prandtl numbers. Other relations should be investigated, even those where σ varies with radius, since a digital computer can handle this variation.

5. Finally, experimental verification is required for the problems of combined forced and free turbulent convection in a vertical tube.

ACKNOWLEDGMENTS

The authors acknowledge the helpful discussions held with Dr. R. J. Schoenhals of the School of Mechanical Engineering, Purdue University. In this connection they are also appreciative of the help received from professors, and graduate and undergraduate students of Purdue University.

This work was carried out through the financial assistance of the Argonne National Laboratory, Sub-Contract 31-109-38-704, and the Atomic Energy Commission Contract AT(11-1)-1026.

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Appendix A

NOMENCLATURE

Any consistent set of units may be used. A typical set for engineering calculations is indicated.

A	axial temperature gradient in fluid $\frac{\partial t}{\partial x}$, F/ft
b	distance between two plane parallel surfaces, ft
C	pressure-drop parameter, $-\dfrac{\left(\frac{dp}{dx} + \rho_w \frac{g}{g_c}\right) D^2 g_c}{32\, u_m \mu}$, dimensionless
c_p	specific heat of fluid at constant pressure, $B/lb_m\, F$
D	tube inside diameter, ft
f_x	negative of the X-component of the body force per unit mass, lb_f/lb_m
g	acceleration due to gravity, ft/hr^2
g_c	dimensional constant in Newton's Law, 4.17×10^8 $ft\text{-}lb_m/lb_f\text{-}hr^2$
Gr	Grashof number, $\rho_m \rho_W \beta g\, AD^4/16\mu^2$, dimensionless
h	heat transfer coefficient, $q''_w/-\theta_m$, $B/hr\text{-}ft^2\text{-}F$
J	mechanical equivalent of heat, 778 $ft\text{-}lb_f/B$
k	thermal conductivity of fluid, B/hr-ft-F
Nu	Nusselt number, hD/k, dimensionless
p	static fluid pressure, lb_f/ft^2-abs
Pr	Prandtl number, $c_p\mu/k = \nu/\alpha$, dimensionless
q''_w	heat transfer rate per unit area at wall of tube, $B/hr\text{-}ft^2$
r	radial coordinate, (D/2)-y, ft
R	gas constant of a perfect gas, $ft\text{-}lb_f/lb_m\text{-}R$
Ra	Rayleigh number, $\rho_m \rho_W \beta g\, c_p AD^4/16\mu k$, dimensionless
Re	Reynolds number, $D u_m/\nu$, dimensionless
Re^*	friction Reynolds number, Du^*/ν, dimensionless
t	static fluid temperature, °F
Δt_+	dimensionless temperature difference, $(t - t_W)\rho_m c_p u^*/q''_w$
T	absolute temperature, °R
u	fluid velocity parallel to tube axis at radius r, ft/hr
u^*	friction velocity, $\sqrt{\tau_w g_c/\rho_W}$, ft/hr

u^+	dimensionless velocity u/u^*
U	dimensionless velocity, u/u_m
v	specific volume, ft^3/lb_m
x	distance measured along axis of tube upward from start of heated length, ft
y	coordinate measured radially from tube wall, $\frac{1}{2}D - r$, ft
y_+	dimensionless distance from wall, $y u^*/\nu$
α	thermal diffusivity, $k/\rho c_p$, ft^2/hr
β	thermal coefficient of volume expansion, $\frac{1}{v}\left(\frac{\partial v}{\partial t}\right)_p$, $\frac{ft^3}{ft^3\text{-}F}$
ϵ	eddy diffusivity, ft^2/hr
η	dimensionless radius, $2r/D$
θ	radial temperature difference, $t-t_w$, F
μ	dynamic viscosity of fluid, $lb_m/ft\text{-}hr$
ν	kinematic viscosity of fluid, μ/ρ, ft^2/hr
ρ	mass density of fluid, lb_m/ft^3
σ	ratio of eddy diffusivities, ϵ_H/ϵ_M, dimensionless
τ	fluid shearing stress, lb_f/ft^2
ϕ	dimensionless temperature difference, $2k\theta/\rho_m u_m c_p AD^2$
ψ	angular coordinate in cylindrical coordinate system, radians

Subscripts

av	average
H	heat
m	mixed-mean (also called mixing cup or bulk)
M	momentum
o	reference axial position along tube
W	at wall of tube
X	based on the distance X (rather than diameter D)

Superscripts

'	differentiation once with respect to the independent variable, η
"	per unit area

Appendix B

THE EDDY DIFFUSIVITY OF MOMENTUM

The empirical expression which will be used for the ratio of the eddy diffusivity of momentum ϵ_M to kinematic viscosity ν is given as

$$\frac{\epsilon_M}{\nu} = 0.0667\ \mathrm{Re}^*\ (0.5 + \eta^2)(1-\eta^2) \qquad \text{(B-1a)}$$

for

$$0 \le \eta < \left(1 - \frac{10}{\mathrm{Re}^*}\right)$$

$$\frac{\epsilon_M}{\nu} = 0 \qquad \text{(B-1b)}$$

for

$$\left(1 - \frac{10}{\mathrm{Re}^*}\right) \le \eta \le 1 \quad ,$$

which is equation (28a, b) of the ANALYSIS. This equation comes from Reichardt's work,[45,46] allowance being made for a laminar sublayer $0 \le y^+ \le 5$, which corresponds to $[1-(10/\mathrm{Re}^*)] \le \eta \le 1$. It is plotted in Figure B-1 together with the experimental results of Nikuradse,[35] Nunner,[36] Laufer,[26] and Reichardt.[45,46] Some scatter is observed in these results, but Reichardt's equation correlates the data fairly well considering that the results are based upon an experimentally determined velocity profile. The velocity-profile data of 4 investigators may agree very well, but the slopes of the 4 curves may have large disagreements, as noted by Lykoudis in a discussion of Sleicher's paper.[52]

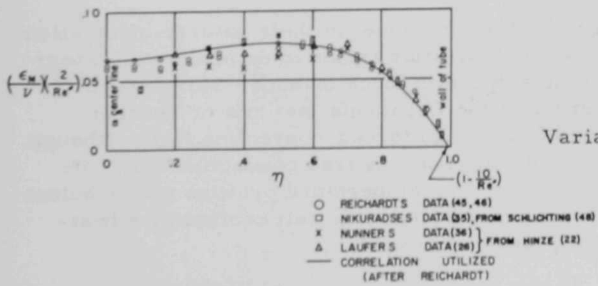

Figure B-1

Variation of Eddy Viscosity in a Tube

There is further evidence, however, that the curve given by the equation (B-1) is of the correct qualitative shape. Corcoran, Opfell, and Sage[7] say that "the eddy viscosity is zero at the wall, but increases rapidly to a maximum roughly midway between the wall and the center of the conduit. This behavior is characteristic of all turbulent flows with an incompressible fluid." Reichardt's empirical equation follows this shape exactly. In addition, the general shape of the curves of Schlinger, Berry, Mason, and Sage,[49] of Page, Schlinger, Breaux, and Sage,[39] and of Page, Corcoran, Schlinger, and Sage[40] agrees with that of equation (B-1). Their maximum value of ϵ_M is about 30% higher than the maximum value given by this equation.

Sherwood and Woertz[50] indicate that the eddy diffusivity is nearly constant in the central section. Their maximum value of ϵ_M/ν is about 50% higher than the maximum value given by equation (B-1). It should be noticed that the curve of this equation is fairly constant (within $\pm$ 10%) over the middle 80% of the tube. This fact also agrees with Clauser's statement[10] that "the very simple assumption of constant eddy viscosity accurately predicts the behavior of the outer 80% to 90% of turbulent (boundary) layers." The experimental work of Wattendorf,[58] who determined turbulent velocity profiles in fully developed Couette flow between concentric cylinders, again indicates that the eddy viscosity is fairly constant in the middle 80% of the space between the walls.

The question arises as to the applicability of equation (B-1) to combined forced and free turbulent convection. All of the experiments cited were presumably carried out in the region of predominantly forced convection. Unfortunately, experiments to determine ϵ_M/ν as a function of radius in free convection or combined forced and free convection could not be found. Yet, except where the flow becomes unstable due to a high negative Rayleigh number [Bosworth[4]], the fact that a condition of turbulent flow exists is the controlling factor.

Holman and Boggs,[23] for instance, in their natural-circulation heat transfer loop, use the usual friction factor to determine the shear friction forces in setting up a dynamic force balance. Murgatroyd[34] used the results of the group at the California Institute of Technology[7,39,40,49] for eddy diffusivity in forced-convection flow, although he was concerned with a problem involving free convection only. He obtained fully developed velocity and temperature profiles for turbulent flow in a long, round vertical or parallel-side cell containing a heat-generating fluid and having isothermal walls.

Townsend[55] considers turbulent flow in a tube to be composed of 2 regions, one near the solid boundary and the other in the central region. The region adjacent to the wall is one "within which the total shear stress is nearly constant and whose motion is determined almost entirely by the shear stress and the fluid viscosity." The motion in the central region is determined by the friction velocity u* and the tube radius. Therefore, since the turbulent motion throughout the tube depends only on the friction velocity, fluid viscosity, and tube radius, it is assumed that there is no difference in the value of ϵ_M/ν in pure forced convection than in either pure free convection or combined forced and free convection. Experimental work will be required to test this assumption.

If later, and better, experimental values of ϵ_M/ν for combined forced and free convection become available, these values can easily be incorporated into the method of solution outlined in the ANALYSIS.

Appendix C

EQUATION OF STATE

If we expand ρ in a Taylor series in t about the reference temperature t_W, we obtain:(3)

$$\rho = \rho\Big|_{t_W} + \frac{\partial \rho}{\partial t}\Big|_{t_W} (t-t_W) + \frac{1}{2}\,\frac{\partial^2 \rho}{\partial t^2}\Big|_{t_W} (t-t_W)^2 + \ldots \qquad \text{(C-1)}$$

But

$$\beta \equiv \frac{1}{v}\left(\frac{\partial v}{\partial t}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t}\right)_p \quad . \qquad \text{(C-2)}$$

Therefore,

$$\rho = \rho_W - \beta\rho_W (t-t_W) + \frac{1}{2}\,\frac{\partial^2 \rho}{\partial t^2}\Big|_{t} (t-t_W)^2 + \ldots \qquad \text{(C-3)}$$

or

$$\rho = \rho_W(1-\beta\theta) \quad , \qquad \text{(C-4)}$$

where θ has been previously defined as $t-t_W$ and only the first 2 terms of the Taylor series have been used. This is equation (5) in the ANALYSIS, which may also be obtained by using finite differences for the differentials in equation (C-2) above. For a gas which obeys the perfect gas law, pv = RT, the equation of state may be written as

$$\rho = \rho \left(1 - \frac{\theta}{T_W}\right) \quad , \qquad \text{(C-5)}$$

where T is the absolute temperature.

Appendix D

THE LONGITUDINAL TEMPERATURE GRADIENT

The temperature of the fluid is a function of x and r in the problem considered. But, for fully developed velocity and temperature profiles, the temperature difference θ is a function of r only:

$$\theta(r) \equiv t\,(x,r) - t\left(x, \frac{D}{2}\right) = t - t_W \quad . \tag{D-1}$$

Now,

$$\frac{\partial \theta(r)}{\partial x} = 0 = \frac{\partial t}{\partial x} - \frac{\partial t_W}{\partial x} \quad , \tag{D-2}$$

or

$$\frac{\partial t_W}{\partial x} = \left(\frac{\partial t}{\partial x}\right)_r \quad . \tag{D-3}$$

Therefore, the temperature gradient in the direction of flow is independent of radial location.

The heat balance for the steady-state condition in our problem states that the rate at which heat is added at the wall is equal to the rate at which heat is taken away by the fluid. In symbolic form (see Figure D-1),

$$q''_w\,(\pi D\,dx) = \left[\rho_m \int_0^{D/2} u\,2\pi r\;dr\right] c_p \frac{\partial t}{\partial x}\,dx \quad , \tag{D-4}$$

from which

$$\frac{\partial t}{\partial x} = \frac{q''_w\,D}{2\,c_p \rho_m \int_0^{D/2} u\,r\,dr} \quad . \tag{D-5}$$

The mean velocity is defined as follows:

$$u_m \equiv \frac{\int_0^{D/2} u\,(2\pi r)\;dr}{\pi D^2/4} = \frac{8\int_0^{D/2} u\,r\,dr}{D^2} \quad . \tag{D-6}$$

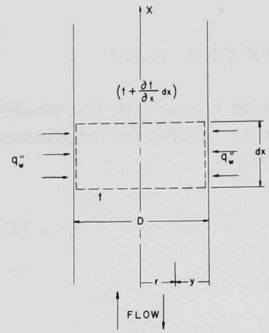

Figure D-1
Heat Balance for an Element of Fluid

Equation (D-5) can therefore be written as

$$\frac{\partial t}{\partial x} = \frac{4 \, q_w''}{c_p \rho_m \, D u_m} \quad . \tag{D-7}$$

Since all quantities on the right-hand side of equation (D-7) are constant in our problem,

$$\frac{\partial t}{\partial x} = A \quad \text{(a constant)} \quad . \tag{D-8}$$

Appendix E

REYNOLDS NUMBER

A dynamic force balance may be set up with the aid of Figure E-1. Equating forces in the downward direction to those in the upward direction ($+ \tau_W$ for upward flow, $- \tau_W$ for downward flow)

$$\pm \tau_W (\pi D dx) + \left(p + \frac{\partial p}{\partial x} dx\right)\left(\frac{\pi}{4} D^2\right) + \rho_m \left(\frac{\pi}{4} D^2 \, dx\right) \frac{g}{g_c} = p\left(\frac{\pi}{4} D^2\right) \quad , \tag{E-1}$$

from which

$$\frac{\partial p}{\partial x} = \frac{dp}{dx} = \mp \frac{4 \tau_W}{D} - \rho_m \frac{g}{g_c} \quad . \tag{E-2}$$

Since

$$C \equiv \frac{-D^2 g_c \left(\frac{dp}{dx} + \rho_W \frac{g}{g_c}\right)}{32 \, u_m \, \mu} \quad , \tag{E-3}$$

from equation (13)

$$C = \frac{-D^2 g_c \left[\mp \frac{4 \tau_W}{D} + \left(\rho_W - \rho_m\right) \frac{g}{g_c}\right]}{32 \, \mu \, u_m} \quad . \tag{E-4}$$

Using

$$\tau_W = u^{*2} \rho_W / g_c \quad ; \quad \rho_W - \rho_m = \rho_W \beta \theta_m \quad ; \quad \theta_m \equiv \rho_m u_m c_p A D^2 \phi_m / 2k$$

and the definitions of Re, Re*, and Ra, we obtain

$$C = \frac{\pm (Re^*)^2}{8 \, Re} - \frac{Ra \, \phi_m}{4} \tag{E-5}$$

or

$$Re = \frac{\pm (Re^*)^2}{8C + 2 Ra \, \phi_m} \quad , \tag{E-6}$$

which is equation (34) in the ANALYSIS.

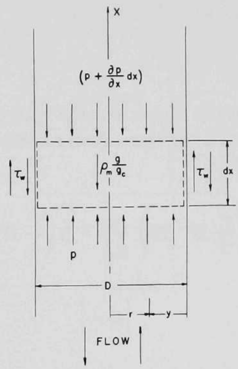

Figure E-1

Forced Acting on an Element of Fluid

It should be noted that the Rayleigh number Ra can be positive or negative. Only the quantities β and A, in the definition of Ra, can change sign. The quantity β is usually positive for most fluids of engineering interest. Water, for instance, has a negative value of β only between 32°F and 39°F. For wall heat addition to the fluid, A is positive for upward flow and negative for downward flow.