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NOMENCLATURE

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Symbol	Description	Defined in Equation
$A_i(\nu)$	Expansion coefficient for region i	(2)
$\mathring{A}_i(\nu)$	Arbitrary even function defined by $A_i(\nu) = \mathring{A}_i(\nu) e^{\lvert y_i \rvert/\ell_i}$	(28)
$\bar{A}_i$	Expansion coefficient	(76)
$B_1(\nu)$	Unknown function of Fredholm integral equation	(38)
$\underline{B}_1(\nu)$	Lower bound for $B_1(\nu)$	(88)
$\bar{B}_1(\nu)$	Second upper bound for $B_1(\nu)$	(89)
$F_1(\nu)$	Nonhomogeneous part of Fredholm equation for region 1	(67)
F_m	Coefficient of Legendre polynomial expansion for $\psi(x,\mu)$	(76)
$G_1(\nu)$	Nonhomogeneous part of Fredholm equation for region 1	(72)
$H_1(\nu,\eta)$	Kernel	(71)
$H(\nu)$	Mean value of $H_1(\nu,\eta)$	(87)
$K_1(\nu,\eta)$	Kernel	(66)
N_i	Normalization constant	(9)
$N_{ik}(\nu)$	Function of ν	(12)
$P_m(\mu)$	Legendre polynomial of order m	(75)
P	Denotes integral to be taken as Cauchy's principal value	(46)
$Q(\eta)$	Arbitrary [illegible]on satisfying Hölder condition	(11)
$R(z)$	Hilbert tr[illegible]n	(45)
$R^{\pm}(\nu)$	Limit valu[illegible] from above and below axis	(46)
$T(z)$	Modified H[illegible]transform	(55)
$T^{\pm}(\nu)$	Limit valu[illegible] from above and below axis	(56)
$a_i^{\pm}$, $\mathring{a}_i$	Arbitrary c[illegible]ts for region i	(2), (28)
c_i, c_k	Number of [illegible]aries per collision for regions i, k	(1), (7)
$f(\zeta)$	Function of [illegible]	(B1)
$f_1(\nu)$	Nonhomogeneous part of singular integral equation for region 1	(43)
$g_1(\nu)$	Factor of $f_1(\nu)$	(44)
ℓ_i	Neutron mean free path for region i	(1)
t_i	Half-thickness in units of mean free path for region i, dimensionless	(37)
u	Parameter	(84)
v	Parameter	(84)
w	Parameter	(84)
x	Distance along coordinate axis perpendicular to plane	(1)
y_i	Half-thickness of region i	(22)
z	Variable	(45)
ϕ	Eigenfunction, from space of solutions of transport equation	(3)
ψ	Neutron density	(1)
$\alpha_1(\nu)$	Auxiliary function	(63)
$\beta_1(\nu,\eta)$	Auxiliary function	(64)
$\gamma_1(\nu,\eta)$	Auxiliary function	(70)
δ	Small parameter	(B2)
$\delta(\nu)$	Delta function	(3)
η	Variable	(7)
λ	Eigenvalue	(65)
$\lambda_i(\nu)$	Proportionality factor of $\delta(\nu)$, defined by $1 - c_i\nu_i \operatorname{arctanh}(\nu) = \lambda_i(\nu)$	(4)
$\lambda_1^{\pm}$	Limit values of $\lambda(z)$	(51a,b)
μ, μ'	Cosine of the angle between neutron velocity and x-axis	(1)
ν	Real parameter	(5)
$\pm\nu_i$	The roots of Eq. (5)	(5)
ζ	Variable	(59)

SPECTRAL REPRESENTATION AND CRITICALITY PROBLEM OF A TWO-REGION CELL TRANSPORT OPERATOR

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I. INTRODUCTION

A two-region reactor cell for slab geometry has been studied in Ref. 1 by the method of spherical harmonics. This paper uses the approach introduced by K. M. Case[2] for the solution of the transport equation to determine accurate conditions for the criticality of a two-region fuel-moderator assembly with reflecting boundaries.

A set of equations developed by A. Kuszell[3] will be used to determine the spectrum of the corresponding transport operator and the corresponding singular eigendistributions for each region. Introduction of appropriate boundary conditions yields a system of integral equations from which the criticality condition and the spectral coefficients for the representation of the vector flux in eigenfunctions can be derived. The derivation of the integral equations is achieved in a new way, namely, by the introduction of an auxiliary function that circumvents the nonanalyticity of a certain function that would occur in the conventional treatment. The results will be compared with those obtained by the method of spherical harmonics for the same problem.

It is assumed that scattering is isotropic and that the fuel-moderator assembly obeys the one-speed Boltzmann equation with constant coefficients:

$$\mu \frac{\partial_i(x,\mu)}{\partial x} + \frac{1}{\ell_i}\psi_i(x,\mu) = \frac{c_i}{2\ell_i}\int_{-1}^{1} \psi_i(x,\mu')d\mu', \tag{1}$$

where the index i denotes the respective region, ψ_i stands for the neutron density, μ is the cosine of the angle between the neutron velocity and the x axis perpendicular to the plane, ℓ_i is the neutron mean free path, which is to be considered constant throughout the region i, and c_i is the number of secondaries per collision, also considered constant throughout the region i.

II. EIGENVALUES AND EIGENDISTRIBUTIONS OF THE BOLTZMANN EQUATION

In accordance with K. M. Case's results, the solution of Eq. (1) can be expressed on each region i as

$$\psi_i(x,\mu) = a_i^+ e^{-x/\ell_i\nu_i}\phi_i(\mu,\nu_i) + a_i^- e^{x/\ell_i\nu_i}\phi_i(\mu,-\nu_i)$$

$$+ \int_{-1}^{1} A_i(\nu) e^{-x/\ell_i\nu}\phi_i(\mu,\nu)\, d\nu, \tag{2}$$

where

$$\phi_i(\mu,\nu) = \frac{c_i}{2}\frac{\nu}{\nu-\mu} + \lambda_i(\nu)\,\delta(\mu-\nu). \tag{3}$$

All integrals involving the term $1/(\nu-\mu)$, where ν and μ are real, are to be understood in the sense of Cauchy's principal value. A rigorous mathematical derivation for the existence of Eq. (2) is given in Ref. 4.

The function $\lambda_i(\nu)$ is given explicitly by

$$\lambda_i(\nu) = 1 - c_i\nu \tanh^{-1}\nu, \tag{4}$$

for the continuous spectrum $-1 \le \nu \le +1$..

The discrete spectrum consists only of $\pm\nu_i$, which are the two roots of

$$c_i\nu_i \tanh^{-1}(1/\nu_i) = 1. \tag{5}$$

If c_i is greater than one, ν_i is purely imaginary. If c_i is less than one, ν_i is real and greater than one. Hence, for the discrete spectrum, the eigenfunctions of Eq. (1) reduce to

$$\Phi_i(\mu,\pm\nu_i) = \frac{c_i}{2}\frac{\nu_i}{\nu_i\pm\mu}. \tag{6}$$

The constants $a_i^{\pm}$ and the functions $A_i(\nu)$ are determined by the given boundary and interface conditions.

III. PROPERTIES OF EIGENFUNCTIONS

A. Kuszell derived the following equations between the eigenfunctions for any regions i and k:

$$\int_{-1}^{1} \mu\phi_i(\mu,\nu)\,\phi_k(\mu,\eta)\,d\mu = \eta\lambda_i(\nu)\,\lambda_k(\eta)\,\delta(\nu-\eta) + \frac{c_i - c_k}{2}\,\frac{\nu\eta}{\nu-\eta}; \tag{7}$$

$$\int_{-1}^{1} \mu\phi_i(\mu,\pm\nu_i)\,\phi_k(\mu,\nu)\,d\mu = \frac{c_i - c_k}{c_i}\,\nu\phi_i(\nu,\pm\nu_i); \tag{8}$$

$$\int_{-1}^{1} \mu\phi_i(\mu,\pm\nu_i)\,\phi_k(\mu,\pm\nu_k)\,d\mu = \begin{cases} \pm\dfrac{c_i - c_k}{2}\,\dfrac{\nu_i\nu_k}{\nu_i - \nu_k} & \text{for } i \neq k, \\[2ex] \pm\dfrac{c_i}{2}\,\nu_i\left[\dfrac{c_i\nu_i^2}{\nu_i^2 - 1} - 1\right] = \pm\nu_i N_i & \text{for } i = k; \end{cases} \tag{9}$$

$$\int_{-1}^{1} \mu\phi_i(\mu,\pm\nu_i)\,\phi_k(\mu,\mp\nu_k)\,d\mu = \pm\frac{c_k - c_i}{2}\,\frac{\nu_i\nu_k}{\nu_i + \nu_k}; \tag{10}$$

$$\int_{-1}^{1} \mu\phi_i(\mu,\pm\nu_i)\,d\mu \int_{-1}^{1} Q(\eta)\,\phi_k(\mu,\eta)\,d\eta = \frac{c_i - c_k}{c_i}\int_{-1}^{1} \eta Q(\eta)\,\phi_i(\eta,\pm\nu_i)\,d\eta; \tag{11}$$

$$\begin{aligned} \int_{-1}^{1} \mu\phi_i(\mu,\nu)\,d\mu \int_{-1}^{1} Q(\eta)\,\phi_k(\mu,\eta)\,d\eta &= \nu\left[\lambda_i(\nu)\,\lambda_k(\nu) + \frac{\pi^2}{4}\,c_i c_k \nu^2\right]Q(\nu) \\ &\quad + \frac{c_k - c_i}{2}\,\nu\int_{-1}^{1} \frac{\eta Q(\eta)\,d\eta}{\eta - \nu} \\ &= \nu N_{ik}(\nu) Q(\nu) + \frac{c_k - c_i}{2}\,\nu\int_{-1}^{1} \frac{\eta Q(\eta)\,d\eta}{\eta - \nu}. \end{aligned} \tag{12}$$

Equations (11) and (12) are valid for any function $Q(\eta)$ that satisfies a Hölder condition.

When $i = k$ in Eqs. (7-10), the orthogonality relations derived by Case are obtained.

These relations are independent of the boundary and interface conditions. To check the consistency of the two equations of Eq. (9) as $c_k \to c_i$, the following relation, which can be easily derived from Eq. (5), is helpful:

$$\frac{dc_i}{d\nu_i}\,\frac{\nu_i}{2} = N_i. \tag{13}$$

In addition, note that

$$\phi_i(-\mu,\pm\nu_i) = \phi_i(\mu,\mp\nu_i), \tag{14}$$

and

$$\phi_i(-\mu,\nu) = \phi_i(\mu,-\nu), \qquad (-1 \le \nu \le 1). \tag{15}$$

With the aid of Eqs. (3) and (6), Kuszell's relations [Eqs. (7-12)], and assuming $\psi_i(x,\mu)$ is given in the form of Eq. (2), we obtain the following set of transformations for $-1 \le \nu \le 1$:

$$\int_{-1}^{1} \mu\phi_i(\mu,\pm\nu_i)\,\psi_i(x,\mu)\,d\mu = \pm\nu_i N_i a_i^{\pm}\, e^{\mp x/\ell_i\nu_i}, \tag{16}$$

$$\int_{-1}^{1} \mu\phi_i(\mu,\nu)\,\psi_i(x,\mu)\,d\mu = \nu N_{ii}(\nu)\,A_i(\nu)\,e^{-x/\ell_i\nu}, \tag{17}$$

$$\int_{-1}^{1} \mu\phi_k(\mu,\pm\nu_k)\,\phi_i(x,\mu)\,d\mu = \pm\nu_k\,\frac{c_i - c_k}{2}\left[\frac{\nu_i}{\nu_i \mp \nu_k}\,a_i^{+}\,e^{-x/\ell_i\nu_i} + \frac{\nu_i}{\nu_i \pm \nu_k}\,a_i^{-}e^{x/\ell_i\nu_i} + \int_{-1}^{1}\frac{\eta}{\eta \mp \nu_k}\,A_i(\eta)\,e^{-x/\ell_i\eta}\,d\eta\right], \tag{18}$$

and

$$\int_{-1}^{1} \mu\phi_k(\mu,\nu)\,\psi_i(x,\mu)\,d\mu = \nu\,\frac{c_i - c_k}{2}\left[\frac{\nu_i}{\nu_i - \nu}\,a_i^{+}\,e^{-x/\ell_i\nu_i} + \frac{\nu_i}{\nu_i + \nu}\,a_i^{-}\,e^{x/\ell_i\nu_i} + \int_{-1}^{1}\frac{\eta}{\eta - \nu}\,A_i(\eta)\,e^{-x/\ell_i\eta}\,d\eta\right] + \nu\,\frac{c_i - c_k}{2}\left[\frac{2N_{ik}(\nu)}{c_i - c_k}\,A_i(\nu)\,e^{-x/\ell_i\nu}\right]. \tag{19}$$

The consistency of Eqs. (16) and (18) when $c_k \to c_i$ can easily be established by Eq. (13). A similar consistency exists between Eqs. (17) and (19).

We can write Eqs. (18) and (19) as follows:

$$\int_{-1}^{1} \mu\phi_k(\mu,\pm\nu_k)\, \psi_i(x,\mu)\, d\mu = \pm\nu_k \frac{c_i - c_k}{c_i} \psi_i(x,\pm\nu_k), \tag{20}$$

$$\int_{-1}^{1} \mu\phi_k(\mu,\nu)\, \psi_i(x,\mu)\, d\mu = \nu \frac{c_i - c_k}{c_i}\left[\psi_i(x,\nu) + \left\{\frac{c_i}{c_i - c_k} N_{ik}(\nu) - \lambda_i(\nu)\right\} A_i(\nu)\, e^{-x/\ell_i\nu}\right], \tag{21}$$

where ψ_i can be considered an analytic extension of the function $\psi_i(x,\mu)$ as given by Eq. (2) into the complex μ plane. Hence, it will have only symbolic, but no physical, meaning.

These last equations will be used to obtain a formulation of the critical problem and to determine the expansion coefficients in Eq. (2).

IV. THE CRITICAL PROBLEM OF A TWO-REGION SLAB WITH REFLECTING BOUNDARIES

We consider a two-region infinite slab of a multiplying ($c > 1.0$) and an absorbing ($c < 1.0$) medium, a simplification of a fuel rod consisting of uranium plates immersed in water. We restrict the discussion to a single cell as illustrated in Fig. 1.

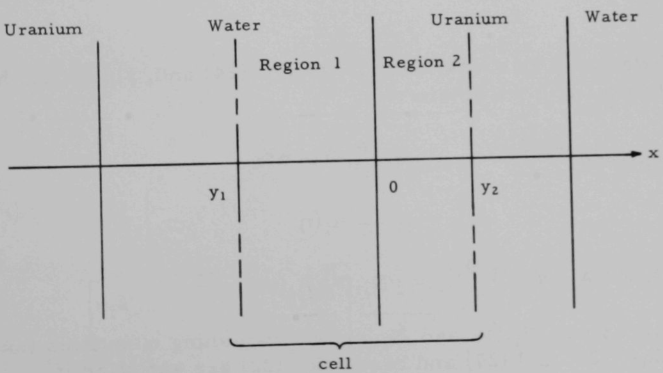

Fig. 1. Reactor Cell and Coordinate Axis Representation

The symmetry of the problem suggests a coordinate axis perpendicular to the plane interface with its origin at one of the interfaces. The half-thicknesses of the regions are $-y_1$ and y_2; the material constants of the regions are c_1 and ℓ_1, and c_2 and ℓ_2, respectively.

The solution of the general periodic problem seemed quite cumbersome, and therefore this paper is restricted to the two-region slab with reflecting boundaries.

On the reflecting boundaries, we impose the condition

$$\psi_i(y_i,\mu) = \psi_i(y_i,-\mu) \tag{22}$$

for $i = 1,2$. Because of continuity, the interface condition is

$$\psi_1(0,\mu) = \psi_2(0,\mu). \tag{23}$$

Eq. (22) implies periodicity.

Because we have a complete set of eigenfunctions, Eq. (22) can be written in equivalent forms:

$$\int_{-1}^{1} \mu\phi_i(\mu_1 \pm \nu_i)\, \psi_i(y_i,\mu)\, d\mu = \int_{-1}^{1} \mu\phi_i(\mu,\pm\nu_i)\, \psi_i(y_i,-\mu)\, d\mu, \tag{24}$$

and

$$\int_{-1}^{1} \mu\phi_i(\mu,\nu)\, \psi_i(y_i,\mu)\, d\mu = \int_{-1}^{1} \mu\phi_i(\mu,\nu)\, \psi_i(y_i,-\mu)\, d\mu, \tag{25}$$

for $i = 1,2$.

If we apply Eqs. (14) and (16) to Eq. (24) and, similarly, Eqs. (15) and (17) to Eq. (25), we obtain

$$\pm\nu_i N_i a_i^{\pm} e^{\mp y_i/\ell_i\nu_i} = \pm\nu_i N_i a_i^{\mp} e^{\pm y_i/\ell_i\nu_i}, \tag{26}$$

and

$$\nu N_{ii}(\nu)\, A_i(\nu)\, e^{-y_i/\ell_i\nu} = \nu N_{ii}(\nu)\, A_i(-\nu)\, e^{y_i/\ell_i\nu}, \tag{27}$$

for $i = 1,2$. Since $N_{ii}(\nu)$ and N_i are nonvanishing quantities (except when $c_i = 0$), Eqs. (26) and (27) and hence Eq. (22) are satisfied if

$$\left.\begin{aligned} a_i^{\pm} &= \mathring{a}_i e^{\pm y_i/\ell_i\nu_i}, \\ A_i(\nu) &= \mathring{A}(\nu)\, e^{y_i/\ell_i\nu}, \quad i = 1,2, \end{aligned}\right\} \tag{28}$$

where $\mathring{a}_i$ is an arbitrary constant and $\mathring{A}_i(\nu)$ is an arbitrary even function in ν, which have to be determined.

We now apply transformations on the interface condition, Eq. (23), similar to those used for the boundary conditions, Eq. (22). Again, because we have a complete set of eigenfunctions, Eq. (23) can be written in equivalent forms:

$$\int_{-1}^{1} \mu\phi_2(\mu_1 \pm \nu_2)\, \psi_1(0,\mu)\, d\mu = \int_{-1}^{1} \mu\phi_2(\mu_1 \pm \nu_2)\psi_2(0,\mu)\, d\mu, \tag{29}$$

and

$$\int_{-1}^{1} \mu\phi_2(\mu,\nu)\, \psi_1(0,\mu)\, d\mu = \int_{-1}^{1} \mu\phi_2(\mu,\nu)\, \psi_2(0,\mu)\, d\mu. \tag{30}$$

If we apply Eq. (20) to the left-hand side and Eq. (16) to the right-hand side of Eq. (29), and similarly apply Eqs. (21) and (17) to Eq. (30), and then make the substitutions of Eq. (28), we obtain

$$\frac{c_1 - c_2}{c_1}\, \psi_1(0,\pm\nu_2) = N_2 \mathring{a}_2 e^{\pm y_2/\ell_2\nu_2}, \tag{31}$$

and

$$\frac{c_1 - c_2}{c_1}\left[\psi_1(0,\nu) + \left\{\frac{c_1}{c_1 - c_2} N_{12}(\nu) - \lambda_1(\nu)\right\}\mathring{A}_1(\nu)\, e^{y_1/\ell_1\nu}\right] = N_{22}(\nu)\mathring{A}_2(\nu)\, e^{y_2/\ell_2\nu}, \tag{32}$$

where

$$-1 \le \nu \le 1.$$

A similar set of equations can be obtained by interchanging the indices one and two.

V. REDUCTION OF THE SYSTEM TO A SINGULAR INTEGRAL EQUATION WITH ONLY ONE UNKNOWN FUNCTION AND ONE COMPATIBILITY CONDITION

The purpose of this section is to eliminate the unknown constant $\mathring{a}_2$ and function $\mathring{A}_2(\nu)$ in Eqs. (31) and (32) and rewrite the equations in dimensionless form.

By multiplying the two expressions of Eq. (31) by $e^{\mp y_2/\ell_2\nu_2}$, respectively, and subtracting, we obtain

$$\psi_1(0,\nu_2)\; e^{-y_2/\ell_2\nu_2} - \psi_1(0,-\nu_2)\; e^{y_2/\ell_2\nu_2} = 0. \tag{33}$$

In a similar manner, we eliminate $\mathring{A}_2(\nu)$ from Eq. (32). We rewrite Eq. (32) as

$$\psi_1(0,\nu)\; e^{-y_2/\ell_2\nu} + \Big\{[c_1/(c_1 - c_2)]\, N_{12}(\nu) - \lambda_1(\nu)\Big\}$$
$$\mathring{A}_1(\nu)\; e^{y_1/\ell_1\nu - y_2/\ell_2\nu} = [c_1/(c_1 - c_2)]\, N_{22}(\nu)\; \mathring{A}_2(\nu).$$

If we replace ν by $-\nu$ in this equation and take the difference of the two, we obtain, upon noting that $N_{12}(\nu)$, $\lambda_1(\nu)$, and $\mathring{A}_1(\nu)$ are even functions of ν,

$$\psi_1(0,\nu)\; e^{-y_2/\ell_2\nu} - \psi_1(0,-\nu)\; e^{y_2/\ell_2\nu} + 2\Big\{[c_1/(c_1 - c_2)]\, N_{12}(\nu) - \lambda_1(\nu)\Big\}$$
$$\sinh\Big(y_1/\ell_1\nu - y_2/\ell_2\nu\Big)\; \mathring{A}_1(\nu) = 0. \tag{34}$$

Equations (33) and (34) may be expressed in explicit form by elementary algebraic manipulations. They are, respectively,

$$\nu_1\mathring{a}_1\left[\frac{\sinh(y_1/\ell_1\nu_1 - y_2/\ell_2\nu_2)}{\nu_1 - \nu_2} - \frac{\sinh(y_1/\ell_1\nu_1 + y_2/\ell_2\nu_2)}{\nu_1 + \nu_2}\right]$$
$$+ \int_{-1}^{1} \eta\mathring{A}_1(\eta)\; \frac{\sinh(y_1/\ell_1\eta - y_2/\ell_2\nu_2)}{\eta - \nu_2}\, d\eta = 0, \tag{35}$$

and

$$\nu_1\mathring{a}_1\left[\frac{\sinh(y_1/\ell_1\nu_1 - y_2/\ell_2\nu)}{\nu_1 - \nu} - \frac{\sinh(y_1/\ell_1\nu_1 + y_2/\ell_2\nu)}{\nu_1 + \nu}\right]$$
$$+ [2N_{12}(\nu)/(c_1 - c_2)]\; \mathring{A}_1(\nu)\, \sinh\,(y_1/\ell_1\nu - y_2/\ell_2\nu)$$
$$+ \int_{-1}^{1} \eta\mathring{A}_1(\eta)\; \frac{\sinh(y_1/\ell_1\eta - y_2/\ell_2\nu)}{\eta - \nu}\, d\eta = 0. \tag{36}$$

A similar set of equations for $\mathring{A}_2(\nu)$ can be obtained by interchanging the indices one and two.

The parameters can be made dimensionless by introducing the half-thickness of the region in units of mean free paths:

$$t_1 = -y_1/\ell_1,$$
$$t_2 = y_2/\ell_2. \tag{37}$$

We also introduce the function $B_1(\nu)$ defined as:

$$\mathring{a}_1 B_1(\nu) = \mathring{A}_1(\nu) \cosh(t_1/\nu). \tag{38}$$

Hence, Eqs. (35) and (36) are, respectively,

$$\nu_1\left[\frac{\sinh(t_1/\nu_1 + t_2/\nu_2)}{\nu_1 - \nu_2} - \frac{\sinh(t_1/\nu_1 - t_2/\nu_2)}{\nu_1 + \nu_2}\right]$$
$$+ \int_{-1}^{1} \frac{\eta}{\eta - \nu_2} \frac{\sinh(t_1/\eta + t_2/\nu_2)}{\cosh(t_1/\eta)} B_1(\eta)\, d\eta = 0, \tag{39}$$

and

$$\nu_1\left[\frac{\sinh(t_1/\nu_1 + t_2/\nu)}{\nu_1 - \nu} - \frac{\sinh(t_1/\nu_1 - t_2/\nu)}{\nu_1 + \nu}\right] + \frac{2N_{12}(\nu)}{c_1 - c_2} \frac{\sinh(t_1/\nu + t_2/\nu)}{\cosh(t_1/\nu)} B_1(\nu)$$
$$+ \int_{-1}^{1} \frac{\eta}{\eta - \nu} \frac{\sinh(t_1/\eta + t_2/\nu)}{\cosh(t_1/\eta)} B_1(\eta)\, d\eta = 0. \tag{40}$$

Equation (40) is a singular integral euqation for the unknown function $B_1(\nu)$ with the compatibility condition Eq. (39), which we call the "criticality condition."

We write

$$\frac{\sinh(t_1/\eta + t_2/\nu)}{\cosh(t_1/\eta)} = \frac{\sinh(t_1/\nu + t_2/\nu)}{\cosh(t_1/\nu)} + \frac{\sinh(t_1/\eta + t_2/\nu)}{\cosh(t_1/\eta)} - \frac{\sinh(t_1/\nu + t_2/\nu)}{\cosh(t_1/\nu)}$$
$$= \frac{\sinh[(t_1 + t_2)/\nu]}{\cosh(t_1/\nu)} + [\tanh(t_1/\eta) - \tanh(t_1/\nu)] \cosh(t_2/\nu). \tag{41}$$

If we insert Eq. (41) under the integral in Eq. (40) and rearrange terms, we obtain

$$N_{12}(\nu)\, B_1(\nu) + \frac{c_1 - c_2}{2} \int_{-1}^{1} \frac{\eta}{\eta - \nu} B_1(\eta)\, d\eta = f_1(\nu), \tag{42}$$

where

$$f_1(\nu) = -\frac{c_1 - c_2}{2} \frac{\cosh(t_1/\nu)}{\sinh[(t_1 + t_2)/\nu]} g_1(\nu), \tag{43}$$

and

$$g_1(\nu) = \nu_1 \left[\frac{\sinh(t_1/\nu_1 + t_2/\nu)}{\nu_1 - \nu} - \frac{\sinh(t_1/\nu_1 - t_2/\nu)}{\nu_1 + \nu} \right]$$

$$+ \cosh \frac{t_2}{\nu} \int_{-1}^{1} \frac{\tanh(t_1/\eta) - \tanh(t_1/\nu)}{\eta - \nu} \eta B_1(\eta)\, d\eta. \tag{44}$$

VI. TRANSFORMATION INTO A FREDHOLM INTEGRAL EQUATION

Equation (42) is a singular integral equation for $B_1(\nu)$, which we shall transform into an equivalent Fredholm equation of the second kind. The method consists in applying the Hilbert transformation on $\nu B_1(\nu)$ and reducing Eq. (42) into a Hilbert problem. The solution of the Hilbert problem furnishes an integral equation of the second kind for $B_1(\nu)$.

We set

$$R(z) = \frac{c_1 - c_2}{2\pi i} \int_{-1}^{1} \frac{\eta B_1(\eta)}{\eta - z}\, d\eta. \tag{45}$$

This choice is prompted by the ease of comparison with the method of Case. The function R has the following properties:

a) It is analytic in the complex plane with cut from -1 to +1.

b) It vanishes at infinity as $1/z$.

The limit values of R as z approaches the cut are given by Muskhelishvili.[(5)] These are known as Plemelj's formula:

$$R^{\pm}(\nu) = \pm \frac{c_1 - c_2}{2} \nu B_1(\nu) + \frac{c_1 - c_2}{2\pi i} P \int_{-1}^{1} \frac{\eta B_1(\eta)\, d\eta}{\eta - \nu}, \tag{46}$$

where P denotes that the integral is to be taken as Cauchy's principal value. Adding and subtracting these expressions, we obtain

$$R^{+}(\nu) + R^{-}(\nu) = \frac{c_1 - c_2}{\pi i} P \int_{-1}^{1} \frac{\eta B_1(\eta)\, d\eta}{\eta - \nu}, \tag{47}$$

and

$$R^{+}(\nu) - R^{-}(\nu) = (c_1 - c_2)\, \nu B_1(\nu). \tag{48}$$

We now substitute these equations into Eq. (42), collect similar terms, and obtain

$$[N_{12} + (c_1 - c_2)\, \pi i \nu/2]\, R^{+} - [N_{12} - (c_1 - c_2)\, \pi i \nu/2]\, R^{-} = (c_1 - c_2)\, \nu f_1(\nu) \tag{49}$$

The coefficients of $R^{\pm}$ can be written as

$$\lambda_1^{+}(\nu)\, \lambda_2^{-}(\nu) = N_{12} + (c_1 - c_2)\, \pi i \nu/2, \tag{50}$$

and

$$\lambda_1^{-}(\nu)\, \lambda_2^{+}(\nu) = N_{12} - (c_1 - c_2)\, \pi i \nu/2. \tag{51}$$

The validity of these equations can be shown readily. We set

$$\lambda_i(z) = 1 + \frac{c_i}{2} z \int_{-1}^{1} \frac{d\eta}{\eta - z} = 1 - c_i z \tanh^{-1} \frac{1}{z}. \tag{51a}$$

Then, by Plemelj's formula,

$$\lambda_i^{\pm}(\nu) = 1 + \frac{c_i}{2} \nu P \int_{-1}^{1} \frac{d\eta}{\eta - \nu} \pm \frac{\pi i}{2} c_i \nu. \tag{51b}$$

But by definition,

$$\lambda_i(\nu) = 1 + \frac{c_i}{2} \nu P \int_{-1}^{1} \frac{d\eta}{\eta - \nu},$$

and

$$N_{ik} = \lambda_i(\nu)\,\lambda_k(\nu) + (\pi^2/4)\,c_i c_k \nu^2.$$

Hence, Eqs. (50) and (51) follow immediately if one considers that

$$\lambda_1 c_2 - \lambda_2 c_1 = c_2 - c_1.$$

Similarly, one obtains

$$N_{ii}(\nu) = \lambda_i^+(\nu)\,\lambda_i^-(\nu). \tag{52}$$

We can now write Eq. (42) as

$$\lambda_1^+\lambda_2^- R^+ - \lambda_1^-\lambda_2^+ R^- = (c_1 - c_2)\,\nu f_1(\nu). \tag{53}$$

If this equation is divided by $\lambda_2^+\lambda_2^- = N_{22}$, we obtain

$$\frac{\lambda_1^+ R^+}{\lambda_2^+} - \frac{\lambda_1^- R^-}{\lambda_2^-} = \frac{c_1 - c_2}{N_{22}}\,\nu f_1(\nu). \tag{54}$$

Since the function $\lambda_1(z)/\lambda_2(z)$, or its inverse, is not analytic in the complex plane cut from -1 to +1, we introduce the auxiliary function

$$T(z) = \frac{\nu_2^2 - z^2}{\nu_1^2 - z^2}\,\frac{\lambda_1(z)}{\lambda_2(z)}\,R(z), \tag{55}$$

where $\pm\nu_1$ and $\pm\nu_2$ are the only zeros of $\lambda_1(z)$ and $\lambda_2(z)$, respectively. Hence, we can write Eq. (53) as

$$T^+(\nu) - T^-(\nu) = \frac{\nu_2^2 - \nu^2}{\nu_1^2 - \nu^2}\,\frac{c_1 - c_2}{N_{22}(\nu)}\,\nu f_1(\nu). \tag{56}$$

From this, we obtain

$$T(z) = \frac{c_1 - c_2}{2\pi i}\int_{-1}^{1} \frac{\nu_2^2 - \zeta^2}{\nu_1^2 - \zeta^2}\,\frac{f_1(\zeta)}{N_{22}(\zeta)}\,\frac{\zeta}{\zeta - z}\,d\zeta, \tag{57}$$

and hence,

$$R(z) = \frac{\nu_1^2 - z^2}{\nu_2^2 - z^2}\,\frac{\lambda_2(z)}{\lambda_1(z)}\,T(z). \tag{58}$$

The function $R^{\pm}$ can then be written as

$$R^{\pm}(\nu) = \pm \frac{c_1 - c_2}{2} \frac{\lambda_2^{\pm}}{\lambda_1^{\pm}} \frac{\nu f_1(\nu)}{N_{22}(\nu)} + \frac{\nu_1^2 - \nu^2}{\nu_2^2 - \nu^2} \frac{\lambda_2^{\pm}}{\lambda_1^{\pm}} \frac{(c_1 - c_2)}{2\pi i} \int_{-1}^{1} \frac{\nu_2^2 - \zeta^2}{\nu_1^2 - \zeta^2} \frac{f_1(\zeta)}{N_{22}(\zeta)} \frac{\zeta}{\zeta - \nu} d\zeta. \tag{59}$$

By Eqs. (47) and (48),

$$R^+ - R^- = (c_1 - c_2)\, \nu B_1(\nu).$$

Substituting the right-hand side of Eq. (59) into the last equation, collecting similar terms, simplifying, and using Eqs. (50), (51), and (52), we obtain

$$B_1(\nu) = \frac{N_{12}(\nu)}{N_{11}(\nu)\, N_{22}(\nu)} f_1(\nu) - \frac{c_1 - c_2}{2N_{11}(\nu)} \frac{\nu_1^2 - \nu^2}{\nu_2^2 - \nu^2} \int_{-1}^{1} \frac{\nu_2^2 - \zeta^2}{\nu_1^2 - \zeta^2} \frac{f_1(\zeta)}{N_{22}(\zeta)} \frac{\zeta}{\zeta - \nu} d\zeta. \tag{60}$$

Since $f_1(\nu)$ is an even function, we can write Eq. (60) in an equivalent form:

$$B_1(\nu) = \frac{N_{12}}{N_{11}N_{22}} f_1 - \frac{c_1 - c_2}{N_{11}} \frac{\nu_1^2 - \nu^2}{\nu_2^2 - \nu^2} \int_{0}^{1} \frac{\nu_2^2 - \zeta^2}{\nu_1^2 - \zeta^2} \frac{f_1(\zeta)}{N_{22}(\zeta)} \frac{\zeta^2}{\zeta^2 - \nu^2} d\zeta. \tag{61}$$

This is a Fredholm integral equation of the second kind for $B_1(\nu)$. We give Eq. (60) in explicit form. To this end, we write Eqs. (43) and (44) as

$$f_1(\nu) = (c_1 - c_2)\left[\alpha_1(\nu) + \int_{-1}^{1} \beta_1(\nu,\eta)\, B_1(\eta)\, d\eta\right], \tag{62}$$

where

$$\alpha_1(\nu) = -\frac{\nu_1}{2} \frac{\cosh(t_1/\nu)}{\sinh[(t_1 + t_2)/\nu]}\left[\frac{\sinh(t_1/\nu_1 + t_2/\nu)}{\nu_1 - \nu} - \frac{\sinh(t_1/\nu_1 - t_2/\nu)}{\nu_1 + \nu}\right], \tag{63}$$

and

$$\beta_1(\nu,\eta) = -\frac{\eta}{2} \frac{\cosh(t_1/\nu)\cosh(t_2/\nu)}{\sinh[(t_1 + t_2)/\nu]} \frac{\tanh(t_1/\eta) - \tanh(t_1/\nu)}{\eta - \nu}. \tag{64}$$

Upon setting

$$\lambda = c_1 - c_2, \tag{65}$$

$$K_1(\nu,\eta) = \frac{N_{12}(\nu)}{N_{11}(\nu)\, N_{22}(\nu)}\, \beta_1(\nu,\eta) - \frac{\lambda}{2}\int_{-1}^{1} \frac{\nu_1^2 - \nu^2}{\nu_2^2 - \nu^2}\, \frac{\nu_2^2 - \zeta^2}{\nu_1^2 - \zeta^2}\, \frac{\beta_1(\zeta,\eta)}{N_{11}(\nu)\, N_{22}(\zeta)}\, \frac{\zeta}{\zeta - \nu}\, d\zeta, \tag{66}$$

and

$$F_1(\nu) = \lambda\left[\frac{N_{12}(\nu)}{N_{11}(\nu)\, N_{22}(\nu)}\, \alpha_1(\nu) - \frac{\lambda}{2}\int_{-1}^{1} \frac{\nu_1^2 - \nu^2}{\nu_2^2 - \nu^2}\, \frac{\nu_2^2 - \zeta^2}{\nu_1^2 - \zeta^2}\, \frac{\alpha_1(\zeta)}{N_{11}(\nu)\, N_{22}(\zeta)}\, \frac{\zeta d\zeta}{\zeta - \nu}\right], \tag{67}$$

Eq. (60) becomes

$$B_1(\nu) - \lambda \int_{-1}^{1} K_1(\nu,\eta)\, B_1(\eta)\, d\eta = F_1(\nu), \tag{68}$$

where the kernel $K_1(\nu,\eta)$ is bounded. The interchange of integration to obtain $K_1(\nu,\eta)$ is legitimate, although the integrals in Eqs. (66) and (67) are to be understood in the sense of Cauchy's principal value. Explicit equations for $K_1(\nu,\eta)$ and $F_1(\nu)$ are derived in Appendix A.

To give Eq. (61) in explicit form, we write Eq. (62) as

$$f_1(\nu) = \lambda[\alpha_1(\nu) + \int_0^1 \gamma_1(\nu,\eta)\, B_1(\eta)\, d\eta], \tag{69}$$

where

$$\gamma_1(\nu,\eta) = -\eta\, \frac{\cosh(t_1/\nu)\cosh(t_2/\nu)}{\sinh[(t_1+t_2)/\nu]}\, \frac{\nu\tanh(t_1/\eta) - \eta\tanh(t_1/\nu)}{\eta^2 - \nu^2}, \tag{70}$$

and $\alpha_1(\nu)$ and λ are given by Eqs. (63) and (65), respectively. The kernel of the integral equation becomes

$$H_1(\nu,\eta) = \frac{N_{12}(\nu)}{N_{11}(\nu)\, N_{22}(\nu)}\, \gamma_1(\nu,\eta) - \lambda\int_{0}^{1} \frac{\nu_1^2 - \nu^2}{\nu_2^2 - \nu^2}\, \frac{\nu_2^2 - \zeta^2}{\nu_1^2 - \zeta^2}\, \frac{\gamma_1(\zeta,\eta)}{N_{11}(\nu)\, N_{22}(\zeta)}\, \frac{\zeta^2}{\zeta^2 - \nu^2}\, d\zeta, \tag{71}$$

and $F_1(\nu)$ becomes

$$G_1(\nu) = \lambda\left[\frac{N_{12}(\nu)}{N_{11}(\nu)\, N_{22}(\nu)}\, \alpha_1(\nu) - \lambda\int_{0}^{1} \frac{\nu_1^2 - \nu^2}{\nu_2^2 - \nu^2}\, \frac{\nu_2^2 - \zeta^2}{\nu_1^2 - \zeta^2}\, \frac{\alpha_1(\zeta)}{N_{11}(\nu)\, N_{22}(\zeta)}\, \frac{\zeta^2}{\zeta^2 - \nu^2}\, d\zeta\right]. \tag{72}$$

Hence Eq. (61) can be written as

$$B_1(\nu) - \lambda \int_0^1 H_1(\nu,\eta)\, B_1(\eta)\, d\eta = G_1(\nu). \tag{73}$$

The remarks after Eq. (68) apply here as well.

VII. REFORMULATION OF CRITICALITY CONDITIONS FOR $c_1 < 1$ AND $c_2 > 1$ AND SUGGESTED USE OF EQUATIONS

For computational purposes, one prefers to deal with real numbers. If we restrict c_1 to a value less than one, and c_2 to a value greater than one, then after some algebraic manipulations, Eq. (39) becomes

$$\frac{\nu_1}{\nu_1^2 + |\nu_2|^2}\left[|\nu_2| \sinh\frac{t_1}{\nu_1} \cos\frac{t_2}{|\nu_2|} - \nu_1 \cosh\frac{t_1}{\nu_1} \sin\frac{t_2}{|\nu_2|}\right] =$$

$$- \int_0^1 \frac{\eta}{\eta^2 + |\nu_2|^2}\left[|\nu_2| \sinh\frac{t_1}{\eta} \cos\frac{t_2}{|\nu_2|} - \eta \sin\frac{t_2}{|\nu_2|} \cosh\frac{t_1}{\eta}\right]\frac{B_1(\eta)}{\cosh(t_1/\eta)}\, d\eta. \tag{74}$$

Equations (73) and (68) already contain only real arguments under these assumptions. Therefore, if we assume values for c_1, c_2, and t_2, we can follow the outline in Table I for finding the value of t_1, which we consider the critical parameter, and for evaluating the flux and criticality condition.

Table I

GENERAL OUTLINE FOR NUMERICAL EVALUATION OF FLUX AND CRITICALITY CONDITION

Step	Operation	Ref. Eq.
1	Determine ν_1 and ν_2. Use Fig. 2	(5)
2	Assume $B_1 = 0$	
3	Determine t_1. Use Fig. 3	(74) and (84)
4	Determine $G_1(\nu)$ or $F_1(\nu)$	(72) or (67)
5	Determine improved $B_1(\nu)$	(73) or (68)
6	Determine improved value of t_1	(74)
7	If t_1 is not sufficiently accurate, repeat steps 4, 5, 6, 7. Otherwise continue with step 8.	
8	Determine $\mathring{A}_1(\nu)/\mathring{a}_1$. Choose convenient $\mathring{a}_1$	(38)
9	Determine $\mathring{a}_2$	(31)
10	Determine $\mathring{A}_2(\nu)$	(32)
11	Determine $a_i^{\pm}$ and $A_i(\nu)$ for $i = 1, 2$	(26, 27)
12	Determine y_1 and y_2	(37)
13	Determine $\psi_i(x, \mu)$ for $i = 1, 2$	(2, 3, 4, 6)

VIII. THE SPECTRUM OF THE P_n-APPROXIMATION

This section will be devoted to a comparison of our method with the well-known method of spherical harmonics.

We assume [see Ref. 1, Eq. (19b)]

$$\psi(x,\mu) = \sum_{m=0}^{\infty} \frac{2m+1}{2} F_m(x)\, P_m(\mu). \tag{75}$$

We can write in the P_n-approximation for $\psi(x,\mu)$,

$$F_m = \sum_{j=0}^{n} \overline{A}_j \nu_j H_m(\nu_j)\, e^{x/\nu_j}, \qquad m = 0, 1, 2, \ldots, n, \tag{76}$$

where the ν_j's are the roots of the characteristic equation

$$\begin{vmatrix}
(1-c)v & 1 & & & & & & & 0 \\
1 & 3v & 2 & & & & & & \\
 & 2 & 5v & 3 & & & & & \\
 & & & \ddots & \ddots & & & & \\
 & & & m & (2m+1)v & m+1 & & & \\
 & & & & & \ddots & \ddots & & \\
 & & & & & & n & (2n+1)v & n+1 \\
0 & & & & & & & \cdot & \cdot
\end{vmatrix} = 0. \tag{77}$$

The $H_m(\nu_j)$'s are determined by the recurrence formulas [see Ref. 1, Eq. (32a)]

$$H_{-1} = 0, \qquad H_0 = 1,$$

$$mH_{m-1} + (2m+1)\,\nu_j H_m + (m+1)\,H_{m+1} = 0, \quad m = 0, 1, 2, \ldots, n-1, \tag{78}$$

and the coefficients $\overline{A}_j$ are determined from the boundary conditions.

Hence, we can write, for the P_n-approximation of Eq. (75),

$$\psi(x,\mu) = \sum_{j=0}^{n} \bar{A}_j \phi_j(\mu,\nu_j)\, e^{x/\nu_j}, \tag{79}$$

where

$$\phi_j = \sum_{m=0}^{n} \frac{2m+1}{2} P_m(\mu)\, \nu_j H_m(\nu_j). \tag{80}$$

This is formally the discrete analog to the representation of Eq. (2) that one would obtain if one expanded eigenfunctions $\phi_j(\mu,\nu)$ in the first n Legendre polynomials, insert them in Eq. (2), and collect similar terms.

The characteristic equation, Eq. (77), was solved with the aid of a computer for two specific values,

$$c_1 = 0.91023923$$

and

$$c_2 = 1.020497037.$$

These values were chosen to coincide with the arbitrary choice of ν_1 and ν_2 used in the numerical example described in Section X. Due to round-off errors, it was necessary to use many significant digits. Table II lists the discrete spectrum for various P_n-approximations for both c_1 and c_2. The value of $\nu_2 = 2.0$ was not obtained until P_5, while the corresponding value of $\nu_1 = 4i$ was obtained in the P_4-approximation as 3.99991i, with no improvement through P_{11}.

Table II

DISCRETE SPECTRUM OF P_n-APPROXIMATION

	Discrete Spectrum, ±	
P_n-approximation, n	$c_2 = 1.020497037$	$c_1 = 0.91023923$
1	4.03259 i	1.92706
2	3.99939 i	1.99505
3	3.99992 i	1.99964
4	3.99991 i	1.99997
5	3.99991 i	2.00000
6	3.99991 i	2.00000
7	3.99991 i	2.00000
8	3.99991 i	2.00000
9	3.99991 i	2.00000
10	3.99991 i	2.00000
11	3.99991 i	2.00000

Tables III and IV give the eigenvalues corresponding to the continuous spectrum for c_1 and c_2. It is evident from the pattern that ever-increasing P_n-approximations are required to span the continuous spectrum $-1 \leq \nu \leq +1$.

Table III

CONTINUOUS SPECTRUM OF P_n-APPROXIMATION FOR c_1 = 0.92023923

Continuous Spectrum	n									
±	11	10	9	8	7	6	5	4	3	2
0.000000		**		**		**		**		**
0.141314	**									
0.172143			**							
0.220028					**					
0.304233							**			
0.306859		**								
0.379513				**						
0.413137	**									
0.488687									**	
0.495178						**				
0.497039			**							
0.585367		**								
0.619943					**					
0.653053	**									
0.701039								**		
0.704899				**						
0.764919			**							
0.807049							**			
0.808691		**								
0.841494	**									
0.866271						**				
0.902312					**					
0.925730				**						
0.941743			**							
0.953148		**								
0.961544	**									

Table IV

CONTINUOUS SPECTRUM OF P_n-APPROXIMATION FOR c_2 = 1.020497037

Continuous Spectrum	n									
±	11	10	9	8	7	6	5	4	3	2
0.000000		**		**		**		**		**
0.143518	**									
0.175402			**							
0.225304					**					
0.311774		**								
0.314057							**			
0.386750				**						
0.418696	**									
0.504548			**							
0.506454						**				
0.511235									**	
0.592413		**								
0.630010					**					
0.659373	**									
0.713245				**						
0.717818								**		
0.771681			**							
0.814150		**								
0.818021							**			
0.845921	**									
0.873523						**				
0.907274					**					
0.929251				**						
0.944324			**							
0.955095		**								
0.963047	**									

IX. CRITICALITY CONDITION FOR P_1-APPROXIMATION AND COMPARISON WITH RESULTS OF SECTION VII

From Eqs. (26) and (27) of Ref. 1, we can read off the criticality condition for the P_1-approximation as

$$(1/\nu_2) \cosh (1/\nu_1)\, t_1 \sinh (1/\nu_2)\, t_2$$

$$+ (1/\nu_1) \cosh (1/\nu_2)\, t_2 \sinh (1/\nu_1)\, t_1 = 0,$$

$$1/\nu_1^2 = 3(1 - c_1), \qquad 1/\nu_2^2 = 3(1 - c_2), \tag{81}$$

where appropriate changes have been made to conform with the notation of this paper. A comparison with the results of Section VII reveals that Eq. (81) is identical to the criticality condition of Eq. (74) if we set $B_1(\nu) \equiv 0$.

If we solve for c_i, we obtain

$$c_i = 1 - (1/\nu_i)^2/3. \tag{82}$$

On the other hand, Eq. (5) yields, upon expansion,

$$c_i = 1 - (1/\nu_i)^2/3 - 4(1/\nu_i)^4/45 - \ldots . \tag{83}$$

Hence, the approximations improve with increasing ν_i or, equivalently, as c_i approaches the value of one.

Equation (74) therefore provides a means of accurately estimating the error of the approximate solution.

X. NUMERICAL EXAMPLE

Let us assume that the values of c_1, c_2, and t_2 are specified. The values of ν_1 and ν_2 can then be determined by Eq. (5) or a set of curves as shown in Fig. 2. If we make the following substitutions in Eq. (74):

$$u = \left| \nu_2/\nu_1 \right|, \qquad v = \left| t_1/\nu_1 \right|, \qquad w = \left| t_2/\nu_2 \right|, \tag{84}$$

and set $B_1(\nu) \equiv 0$, we obtain, as a first-order approximation of the criticality condition,

$$\cosh v \sin w - u \sinh v \cos w = 0, \tag{85}$$

or

$$w = \arctan(u \tanh v). \tag{86}$$

This is a convenient formula for plotting curves of w versus v for the parameter u. A few of these are plotted in Fig. 3. These curves provide a quick way of determining v and consequently t_1.

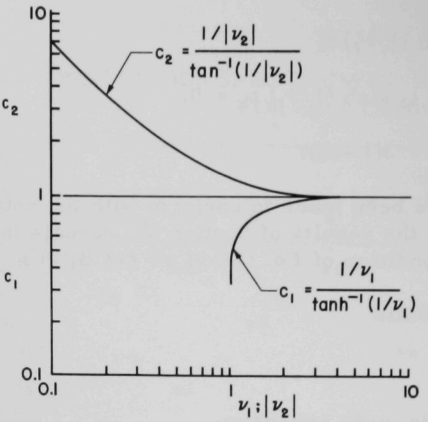

Fig. 2
Discrete Spectrum as a Function of c_i

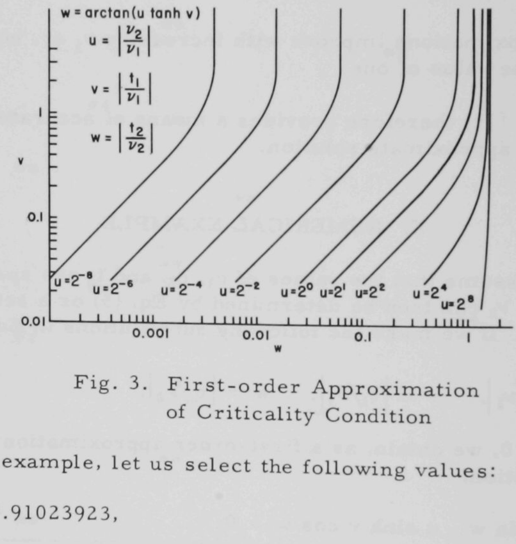

Fig. 3. First-order Approximation of Criticality Condition

As an example, let us select the following values:

$$c_1 = 0.91023923,$$

$$c_2 = 1.020497037,$$

and

$$t_2 = 2.0.$$

We easily determine ν_1 and ν_2 from Eq. (5) to be ± 2.0 and $\pm 4.0i$, respectively. With these values, we find $w = 0.5$, and from Fig. 3 for the curve $u = 2$, we obtain the corresponding value of 0.28 for v. Hence, $t_1 = 0.56$.

With this value of t_1, we may now proceed to calculate α_1, γ_1, $G_1(\nu)$, and $H_1(\nu,\eta)$. It is easy then to calculate $B_1(\nu)$ and make use of Eq. (74) to check the results.

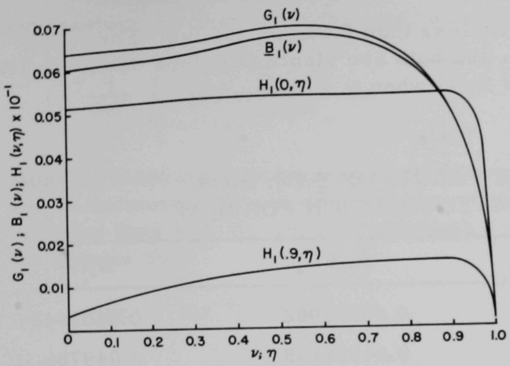

Fig. 4. The Functions $G_1(\nu)$, $B_1(\nu)$, and $H_1(\nu,\eta)$ for Various Values of ν for $t_1 = 0.54$

Alternatively, these values may be determined for a number of values of t_1. Then we plot the left- and right-hand sides of Eq. (74). The intersection determines the exact value of t_1.

Figure 4 represents the plots of $G_1(\nu)$, $H_1(\nu,\eta)$, and $\underline{B}_1(\nu)$, a lower limit for $B_1(\nu)$ for $t_1 = 0.54$. The value of $B_1(\nu)$ was determined by the use of the simplified formula

$$B_1(\nu) \approx \frac{G_1(\nu)}{1 - \lambda H(\nu)}, \tag{87}$$

where

$$H(\nu) = \int_0^1 H_1(\nu,\eta)\, d\eta .$$

This gives a slightly higher value than would be obtained by an iterative procedure. To verify this, let us solve for a lower bound of $B_1(\nu)$. Since $H_1(\nu,\eta)$ and $G_1(\nu)$ are both positive and λ is negative, $B_1(\nu)$ must be positive and consequently less than $G_1(\nu)$. If we substitute $G_1(\nu)$ for $B_1(\nu)$ in Eq. (68), we obtain a lower bound for $B_1(\nu)$. We may then write

$$\underline{B}_1(\nu) = -|\lambda| \int_0^1 H_1(\nu,\eta)\, G_1(\eta)\, d\eta + G_1(\nu). \tag{88}$$

Iterating, we obtain a second upper bound for $B_1(\nu)$ [$G_1(\nu)$ being considered a first upper bound]:

$$\overline{B}_1(\nu) = -|\lambda| \int_0^1 H_1(\nu,\eta)\, \underline{B}_1(\eta)\, d\eta + G_1(\nu). \tag{89}$$

If we take the difference of the two estimates, the result is

$$|\overline{B}_1(\nu) - \underline{B}_1(\nu)| = +|\lambda| \int_0^1 H_1(\nu,\eta) \left[G_1(\eta) - \underline{B}_1(\eta) \right] d\eta .$$

Since $|G_1(\nu) - \underline{B}_1(\nu)| = 0.003785$ and maximum value of H is 0.54, the maximum difference between the upper and lower bounds of $B_1(\nu)$ is

$$|\overline{B}_1(\nu) - \underline{B}_1(\nu)| < 0.11(0.54)\,(0.003785) < \underline{\underline{0.00023}}.$$

The error is consequently less than 0.5% for $\nu < 0.8$. For $\nu > 0.8$, $B_1(\nu)$ approaches $G_1(\nu)$, and both are identically zero when $\nu = 1.0$. Table V compares the values of $B_1(\nu)$ when $t_1 = 0.54$.

Table V

COMPARATIVE VALUES OF $\underline{B}_1(\nu)$, $\overline{B}_1(\nu)$, AND APPROXIMATE SOLUTION FOR $B_1(\nu)$

ν	$\underline{B}_1(\nu)$	$\overline{B}_1(\nu)$	$B_1(\nu)$
0.0	0.06013126	0.06025902	0.0603942
0.9	0.04955709	0.04958148	0.049786

Figure 5 is a plot of the left- and right-hand sides of the "criticality condition" Eq. (74) from which we obtain an exact value of $t_1 = 0.5383$. The first-order approximation, which is equivalent to the P_1-approximation, gives an error of 4%.

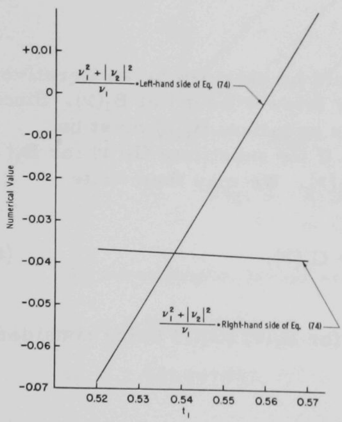

Fig. 5

Numerical Value of Each Side of Criticality Condition as a Function of Criticality Parameter, t_1

XI. CONCLUSIONS

This study has shown that Case's method can be used to determine the flux distribution and criticality condition for a two-region reactor cell with reflecting boundaries. An equation is derived that can be used to determine the accuracy of an approximate solution. A first-order approximation is shown to be equivalent to the P_1-approximation of the spherical harmonic method.

APPENDIX A

Explicit Equations for $K_1(\nu,\eta)$ and $F_1(\nu)$

We consider the integrals in Eqs. (66) and (67) and give explicit expressions for them. To this end, we evaluate first the integral

$$\frac{1}{2\pi i}\int_C g(z)\cdot\frac{\nu_2^2 - z^2}{\lambda_2(z)}\,dz \tag{A1}$$

along a closed contour consisting of a very large circle around the origin, and so deformed that all singularities of g(z) and the interval -1 to +1 are excluded (see Fig. 6); i.e., we establish a domain where g(z) is analytic. We assume that

$$g(z) = 0(z^{-4}) \text{ for } |z| \to \infty,$$

and

$$g^+(z) = g(z)^- \text{ on } (-1,+1).$$

We note that $(\nu_2^2 - z^2)/\lambda_2(z)$ is also analytic in this domain. Hence, the integral (A1) is equal to

the integral over the large circle in the positive direction

+ the sum of the residues of $g(z)\ (\nu_2^2 - z^2)/\lambda_2(z)$

+ the integral around the slit (-1,+1) in the negative direction.

The first term will be negligible if we take the circle large enough. The second term will depend on the special form of g(z).

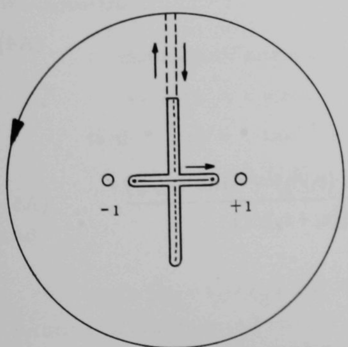

Fig. 6

Singularities of Integrand in (A4)

The integral for the last term is

$$\frac{1}{2\pi i}\int_{-1}^{+1}\left[g(z)\,\frac{\nu_2^2 - z^2}{\lambda_2(z)}\right]^+ dz + \frac{1}{2\pi i}\int_{+1}^{-1}\left[g(z)\,\frac{\nu_2^2 - z^2}{\lambda_2(z)}\right]^- dz =$$

$$\frac{1}{2\pi i}\int_{-1}^{+1} g(z)(\nu_2^2 - z^2)\left[\frac{1}{\lambda_2^+(z)} - \frac{1}{\lambda_2^-(z)}\right] dz =$$

$$-\frac{c_2}{2}\int_{-1}^{+1} g(z)\,\frac{\nu_2^2 - z^2}{N_{22}(z)}\, z dz \tag{A2}$$

because

$$\lambda_2^+(z) - \lambda_2^-(z) = i\pi c_2 z \quad \text{by Plemelj's formula,}$$

and

$$\lambda_2^+(z)\cdot\lambda_2^-(z) = N_{22}(z) \quad \text{by Eq. (52).}$$

Hence, by Cauchy's Theorem,

$$\int_{-1}^{+1} g(\zeta)\,\frac{\nu_2^2 - \zeta^2}{N_{22}(\zeta)}\,\zeta dz = \frac{2}{c_2}\sum \text{Res.}\left\{\frac{g(z)(\nu_2^2 - z^2)}{\lambda_2(z)}\right\}. \tag{A3}$$

To evaluate Eq. (66), we consider the integral

$$\int_{-1}^{+1} \frac{\nu_2^2 - \zeta^2}{\nu_1^2 - \zeta^2}\,\frac{\beta_1(\zeta,\eta)}{N_{22}(\zeta)}\,\frac{\zeta}{\zeta - \nu}\, d\zeta. \tag{A4}$$

We write $\beta_1(\zeta,\eta)$ as follows [see Eq. (64)]:

$$\beta_1(\zeta,\eta) = \frac{\eta}{2}\,\frac{\cosh(t_2/\zeta)}{\eta - \zeta}\,\frac{\cosh(t_1/\zeta)\tanh(t_1/\eta) - \sinh(t_1/\zeta)}{\sinh[(t_1 + t_2)/\zeta]}. \tag{A5}$$

Then,

$$g(z) = \frac{\beta_1(z,\eta)}{(z^2 - \nu_1^2)(\nu - z)}. \tag{A6}$$

The residues of the integrand in Eq. (A4) are

a) from the simple poles where

$\nu_1^2 - z^2 = 0,$

$$r_{\pm\nu_1} = \pm \frac{\beta(\pm\nu_1,\eta)}{2r_1(\nu \mp \nu_1)} \frac{\nu_2^2 - \nu_1^2}{\lambda_2(\nu_1)}; \tag{A7}$$

b) from the simple poles where

$$\sinh \frac{t_1 + t_2}{z} = 0.$$

These are

$$z_k = \frac{t_1 + t_2}{k\pi i}, \qquad k = \pm 1, \pm 2, \pm 3, \ldots . \tag{A8}$$

Hence,

$$r_k = - \frac{\frac{1}{2}\eta \cosh (t_2/z_k)}{(\eta - z_k)(z_k - \nu)} \frac{z_k^2}{t_1 + t_2} \frac{\cosh (t_1/z_k) \tanh (t_1/\eta) \sinh (t_1/z_k)}{\cosh [(t_1 + t_2)/z_k]}$$

$$\frac{\nu_2^2 - z_k^2}{\nu_1^2 - z_k^2} \frac{1}{\lambda(z_k)}. \tag{A9}$$

We note that

$$\begin{aligned} \cosh iz &= \cos z, \\ \sinh iz &= i \sin z, \\ \tanh^{-1} (iz) &= i \tan^{-1} z, \\ \lambda_2(z) &= \lambda_2(-z), \end{aligned}$$

and

$$\cosh [(t_1 + t_2)/z_k] = (-1)^k.$$

We combine r_k and r_{-k} to s_k, which will be real:

$$s_k = r_k + r_{-k}. \tag{A10}$$

After some algebraic manipulations, we obtain

$$s_k = (-1)^{k+1} \frac{|z_k|^2}{t_1 + t_2} \frac{\eta(\nu_2^2 - z_k^2) \cos(t_2/z_k)}{(\eta^2 - z_k)(\nu^2 - z_k^2)(\nu_1^2 - z_k^2)\, \lambda_2(z_k)}$$

$$\left\{(\eta\nu - |z_k|^2) \cos(t_1/|z_k|) \tanh(t_1/\eta) - |z_k|(\eta + \nu) \sin(t_1/|z_k|)\right\}, \tag{A11}$$

with

$$\lambda_2(z_k) = 1 - (c_2/2)\,|z_k| \tan^{-1} |1/z_k|.$$

We do not combine the $r_{\pm\nu_i}$ as in (A10) since in applications one may designate the $\nu_i (i = 1,2)$ such that ν_1 is real. Therefore, the kernel K_1 of Eq. (66) is

$$K_1(\nu,\eta) = \frac{1}{N_{11}(\nu)} \left\{ \frac{N_{12}(\nu)}{N_{22}(\nu)} \beta_1(\nu,\eta) - \frac{c_1 - c_2}{c_2} \frac{\nu_1^2 - \nu^2}{\nu_2^2 - \nu^2} \left[r_{\nu_1} + r_{-\nu_1} + \sum_{k=1}^{\infty} s_k \right] \right\} \tag{66'}$$

Similarly, Eq. (67) yields

$$F_1(\nu) = \frac{c_1 - c_2}{N_{11}(\nu)} \left\{ \frac{N_{12}(\nu)}{N_{22}(\nu)} \alpha_1(\nu) - \frac{1}{c_2} \frac{\nu_1^2 - \nu^2}{\nu_2^2 - \nu^2} \left[\overline{r}_{\nu_1} + \overline{r}_{-\nu_1} + \sum_{k=1}^{\infty} s_k \right] \right\}, \tag{67'}$$

where

$$\overline{r}_{\pm\nu_1} = \pm \frac{\nu_2^2 - y_2^1}{2\nu_1(\nu \mp \nu_1)\, \lambda_2(\nu_1)} \frac{\cosh(t_1/\nu_1) \sinh[(t_1 - t_2)/\nu_1]}{4 \sinh[(t_1 + t_2)/\nu_1]},$$

$$\overline{r}_k = \frac{(\nu_2^2 - z_k^2)}{(z_k - \nu)(\nu_1^2 - z_k^2)\, \lambda_2(z_k)} \frac{\nu_1}{2} \frac{z_k^2}{t_1 + t_2} \frac{\cosh(t_1/z_k)}{\cosh[(t_1 + t_2)/z_k]}$$

$$\left[\frac{\sinh(t_1/\nu_1 + t_2/z_k)}{\nu_1 - z_k} - \frac{\sinh(t_1/\nu_1 - t_2/z_1)}{\nu_1 + z_k} \right],$$

$$z_k = \frac{t_1 + t_2}{k\pi i},$$

$$\bar{s}_k = \bar{r}_k + \bar{r}_{-k},$$

and

$$\bar{s}_k = (-1)^k \frac{2\nu_1 |z_k|^3 (\nu_2^2 - z_k^2)}{(|z_k|^2 + \nu^2)(\nu_1^2 - z_k^2)\, \lambda_2(z_k)}$$

$$\frac{\cos(t_1/z_k)\left[\nu_1 \sin(t_2/|z_k|)\cos(t_1/\nu_1) - |z_k| \sinh(t_1/\nu)\cos(t_2/|z_k|)\right]}{(t_1 + t_2)(\nu_1^2 - z_k^2)}.$$

APPENDIX B

Reduction of Equations for Computer Programming

To facilitate programming for numerical computation, some of the pertinent equations were rewritten. These follow:

$$\alpha_1(\nu) = \frac{[-(\nu/\nu_1 \tanh(t_1/\nu_1) - \tanh(t_2/\nu)] \cosh(t_1/\nu_1)}{[1-(\nu/\nu_1)^2][\tanh(t_1/\nu)+\tanh(t_2/\nu)]}, \qquad \nu \neq 0;$$

$$\alpha_1(0) = -\frac{\cosh t_1/\nu_1}{2}. \tag{63'}$$

$$\gamma_1(\nu,\eta) = \frac{-(\nu/\eta)\tanh(t_1/\eta) + \tanh(t_1/\nu)}{[1-(\nu/\eta)^2][\tanh(t_1/\nu)+\tanh(t_2/\nu)]}, \qquad \eta \neq \nu;$$

$$\gamma_1(\nu,\eta) = \frac{(t_1/\nu)\cosh(t_1/\nu)^{-2} + \tanh(t_1/\nu)}{2[\tanh(t_1/\nu)+\tanh(t_2/\nu)]}, \qquad \eta = \nu;$$

$$\gamma_1(0,\eta) = 0.5;$$

$$\gamma_1(\nu,0) = 0, \ \nu \neq 0. \tag{70'}$$

The function $N_{ik}(\nu)$ defined in Eq. (12) was transformed into

$$N_{ik} = \left[-\frac{c_i\nu}{2}\log\frac{1+\nu}{1-\nu}+1\right]\left[-\frac{c_k\nu}{2}\log\frac{1+\nu}{1-\nu}+1\right] + \pi^2\left(\frac{c_i\nu}{2}\right)\left(\frac{c_k\nu}{2}\right),$$

$$N_{ik}(0) = 1,$$

and

$$N_{ik}(1) = \infty \Longrightarrow G_1(1) = H_1(1,\eta) = 0.$$

Equation (71) was rewritten as

$$H_1(\nu,\eta) = \frac{N_{12}(\nu)}{N_{11}(\nu)\, N_{22}(\nu)}\gamma_1(\nu,\eta)$$

$$-\lambda\frac{\nu_1^2-\nu^2}{\nu_2^2-\nu^2}\frac{1}{N_{11}(\nu)}\int_0^1 \frac{\nu_2^2-\zeta^2}{\nu_1^2-\zeta^2}\frac{\gamma_1(\zeta,\eta)}{N_{22}(\zeta)}\zeta\frac{\zeta}{\zeta^2-\nu^2}\,d\zeta. \tag{71'}$$

The integral in Eq. (71') was calculated by removing the singularity. Upon setting

$$f(\zeta) = \frac{\nu_2^2 - \zeta^2}{\nu_1^2 - \zeta^2} \frac{\gamma_1(\zeta_1 \eta)}{N_{22}} \zeta, \tag{B1}$$

a typical graph of which is shown in Fig. 7, the integral in Eq. (71') can be written as

$$\int_0^1 f(\zeta) \frac{\zeta}{\zeta^2 - \nu^2} d\zeta = \int_0^{\nu-\delta} f(\zeta) \frac{\zeta}{\zeta^2 - \nu^2} d\zeta + \int_{\nu-\delta}^{\nu+\delta} f(\zeta) \frac{\zeta}{\zeta^2 - \nu^2} d\zeta + \int_{\nu+\delta}^{1} f(\zeta) \frac{\zeta}{\zeta^2 - \nu^2} d\zeta. \tag{B2}$$

The first and last integrals on the right-hand side of Eq. (B2) are straightforward and present no difficulties. Since

$$\frac{\zeta}{\zeta^2 - \nu^2} = \frac{1}{2} \frac{1}{\zeta + \nu} + \frac{1}{2} \frac{1}{\zeta - \nu},$$

the middle integral on the right-hand side of Eq. (B2) can be written as

$$\int_{\nu-\delta}^{\nu+\delta} f(\zeta) \frac{\zeta}{\zeta^2 - \nu^2} d\zeta = \frac{1}{2} \int_{\nu-\delta}^{\nu+\delta} \frac{f(\zeta)}{\zeta + \nu} d\zeta + \frac{1}{2} \int_{\nu-\delta}^{\nu+\delta} \frac{f(\zeta)}{\zeta - \nu} d\zeta. \tag{B3}$$

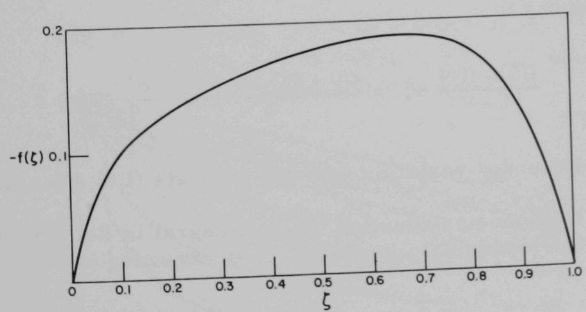

Fig. 7. $f(\zeta)$ as a Function of ζ for $\eta = 0.1$ and $t_1 = 0.56$

Again, the first integral in Eq. (B3) presents no problems. If we subtract and then add $f(\nu)/2(\zeta - \nu)$ to the last integral in Eq. (B3), we obtain

$$\frac{1}{2}\int_{\nu-\delta}^{\nu+\delta} \frac{f(\zeta)}{\zeta - \nu}\, d\zeta = \frac{1}{2}\int_{\nu-\delta}^{\nu+\delta} \frac{f(\zeta) - f(\nu)}{\zeta - \nu}\, d\zeta + \frac{f(\nu)}{2}\int_{\nu-\delta}^{\nu+\delta} \frac{d\zeta}{\zeta - \nu}. \tag{B4}$$

The last integral in Eq. (B4) is zero when taking Cauchy's principal value. The first integral in Eq. (B4) can be reduced by expanding $f(\zeta)$ in a Taylor series about ν; i.e.,

$$\frac{1}{2}\int_{\nu-\delta}^{\nu+\delta} \frac{f(\zeta) - f(\nu)}{\zeta - \nu}\, d\zeta =$$

$$\frac{1}{2}\int_{\nu-\delta}^{\nu+\delta} \left\{ \frac{f(\nu) + (\zeta-\nu)\, f'(\nu) + \frac{(\zeta-\nu)^2\, f''(\nu)}{2!} + \frac{(\zeta-\nu)^3\, f'''(\nu)}{3!} + \ldots - f(\nu)}{\zeta - \nu} \right\} d\zeta$$

$$= \frac{1}{2}\int_{\nu-\delta}^{\nu+\delta} \left\{ f'(\nu) + \frac{(\zeta-\nu)}{2!} f''(\nu) + \frac{(\zeta-\nu)^2}{3!} f'''(\nu) + \ldots \right\} d\zeta$$

$$= \delta f'(\nu) + \frac{\delta^3}{18} f'''(\nu) + \ldots . \tag{B5}$$

Upon using the secant approximation for $f'(\nu)$, i.e.,

$$\delta f'(\nu) = \frac{1}{2}\{f(\nu+\delta) - f(\nu-\delta)\} - \frac{\delta^3}{6} f'''(\nu) + \ldots ,$$

Eq. (B5) reduces to

$$\frac{1}{2}\int_{\nu-\delta}^{\nu+\delta} \frac{f(\zeta) - f(\nu)}{\zeta - \nu}\, d\zeta = \frac{f(\nu+\delta) - f(\nu-\delta)}{2} - \frac{\delta^3}{9} f'''(\nu) + \ldots . \tag{B6}$$

If $f'''(\zeta)$ is of bounded variation, we may truncate (B6) after the first term.

The numerical evaluation of the integral in Eq. (71') was then carried out by trapezoidal integration to an accuracy of 1% in the following form:

$$\int_0^1 f(\zeta)\frac{\zeta}{\zeta^2 - \nu^2}\, d\zeta = \int_0^{\nu-\delta} f(\zeta)\frac{\zeta}{\zeta^2 - \nu^2}\, d\zeta + \int_{\nu+\delta}^1 f(\zeta)\frac{\zeta}{\zeta^2 - \nu^2}\, d\zeta$$

$$+ \int_{\nu-\delta}^{\nu+\delta} \frac{f(\zeta)}{2(\zeta+\nu)}\, d\zeta + \frac{1}{2}[f(\nu+\delta) - f(\nu-\delta)], \tag{B7}$$

where $f(\zeta)$ is given by (B1).

Equation (72) was rewritten as

$$G_1(\nu) = \frac{\lambda N_{12}(\nu)}{N_{11}(\nu)\ N_{22}(\nu)} \alpha_1(\nu)$$

$$-\lambda^2 \frac{\nu_1^2 - \nu^2}{\nu_2^2 - \nu^2} \frac{1}{N_{11}(\nu)} \int_0^1 \frac{\nu_2^2 - \zeta^2}{\nu_1^2 - \zeta^2} \frac{\alpha_1(\zeta)}{N_{22}(\zeta)} \zeta \frac{\zeta}{\zeta^2 - \nu^2} d\zeta. \qquad (72')$$

The integral was evaluated in a manner similar to that of Eq. (71'). Typical graphs for the functions $\alpha_1(\nu)$ and $\gamma_1(\nu, \eta)$ and the integrand of Eq. (72') are shown in Figs. 8 through 11.

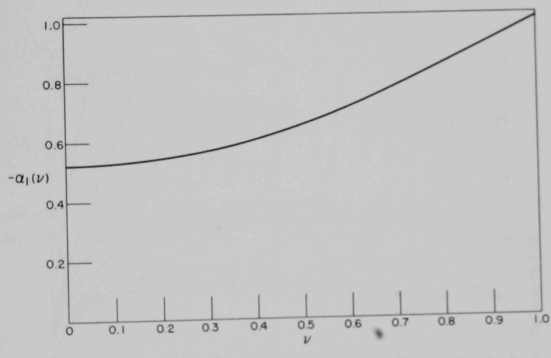

Fig. 8. $\alpha_1(\nu)$, Eq. (63'), as a Function of ν for $t_1 = 0.56$ and $t_2 = 2.0$

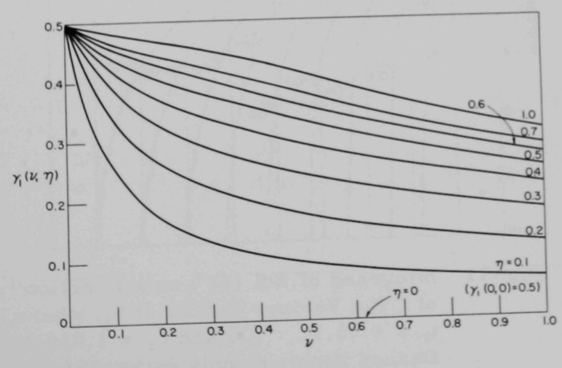

Fig. 9. $\gamma_1(\nu,\eta)$, Eq. (70'), as a Function of ν for Various Values of η, where $t_1 = 0.56$ and $t_2 = 2.0$

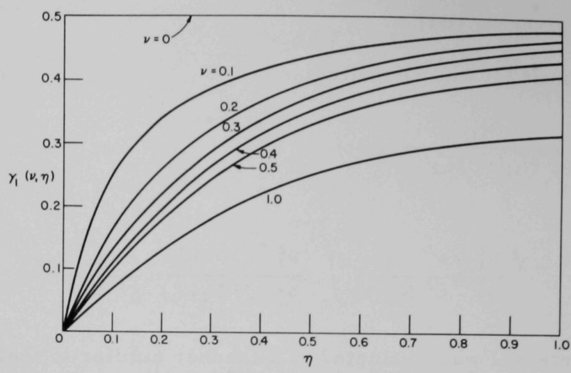

Fig. 10. $\gamma_1(\nu,\eta)$, Eq. (70'), as a Function of η for Various Values of ν, where $t_1 = 0.56$ and $t_2 = 2.0$

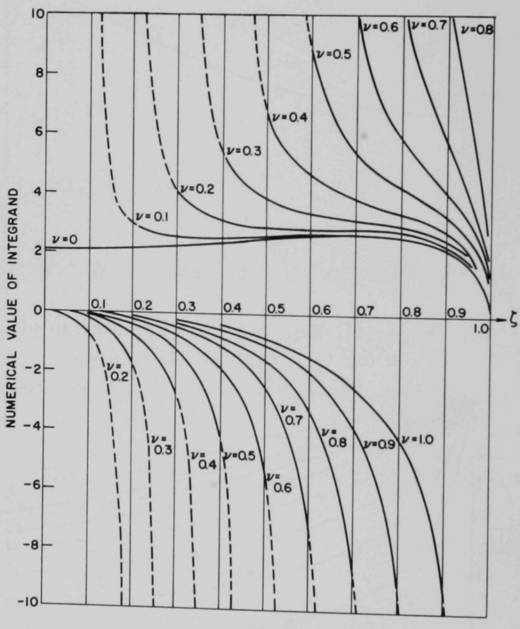

Fig. 11. Integrand of Eq. (72') as a Function of ζ for Various Values at ν_1, where $t_1 = 0.56$, $t_2 = 2.0$ and $c_2 = 1.020$. Dashed lines are only estimates.

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