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Argonne National Laboratory

BENDING OF CIRCULAR PLATES UNDER A VARIABLE SYMMETRICAL LOAD

by

J. C. Heap

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ARGONNE NATIONAL LABORATORY
9700 South Cass Avenue
Argonne, Illinois 60440

BENDING OF CIRCULAR PLATES UNDER A VARIABLE SYMMETRICAL LOAD

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J. C. Heap

Particle Accelerator Division

April 1964

Operated by The University of Chicago
under
Contract W-31-109-eng-38
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NOMENCLATURE

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English Letters	Description	British Units	Metric Units
A	Area	in.^{2}	cm^2
a	Outer plate support and load distribution radius	in.	cm
b	Inner plate and/or load distribution radius	in.	cm
C_1 C_2 C_3	Constants of integration for variable load over entire actual plate or bounded by circles of inner radius and outer plate support-load radius	1/in. in. in.	1/cm cm cm
C_4 C_5 C_6	Constants of integration for inner portion of solid plate, variable load over plate bounded by circles of inner radius and outer plate support-load radius	1/in. in. in.	1/cm cm cm
D	Flexural rigidity of plate, symbolically $Eh^3/12(1 - \nu^2)$	lb$_f$-in.	kg$_f$-cm
E	Modulus of elasticity	lb$_f$/in.^{2}	kg$_f$/cm^2
h	Uniform plate thickness	in.	cm
I	Second moment of area per unit length, symbolically $h^3/12$	in.^{4}/in.	cm^4/cm
k_d	Maximum de [illegible] constant		
k_m	Maximum be [illegible] ment constant		
M_{max}	Maximum be [illegible] ment per unit length	lb$_f$-in./in.	kg$_f$-cm/cm
M_r	Radial bendin [illegible] t per unit length	lb$_f$-in./in.	kg$_f$-cm/cm
M_{ra}	Radial bending [illegible] t per unit length at outer plate [illegible] load radius	lb$_f$-in./in.	kg$_f$-cm/cm
M_{rb}	Radial bending moment per unit length at inner plate and/or load distribution radius	lb$_f$-in./in.	kg$_f$-cm/cm
M_t	Tangential bending moment per unit length	lb$_f$-in./in.	kg$_f$-cm/cm
M_{ta}	Tangential bending moment per unit length at outer plate support-load radius	lb$_f$-in./in.	kg$_f$-cm/cm
M_{tb}	Tangential bending moment per unit length at inner plate and/or load distribution radius	lb$_f$-in./in.	kg$_f$-cm/cm
P	Constant force	lb$_f$	kg$_f$
r	Radius of plate	in.	cm
V	Shearing force per unit circumferential length	lb$_f$/in.	kg$_f$/cm
w	Deflection of plate	in.	cm
y	Distance within plate thickness from middle surface along a normal	in.	cm

Greek Letters	Description	British Units	Metric Units
ϵ_r	Radial unit strain	in./in.	cm/cm
ϵ_t	Tangential unit strain	[illegible]/in.	cm/cm
θ	Tangential angle		rad
ν	Poisson's ratio		
σ_{max}	Maximum unit stress	[illegible]/in.^{2}	kg$_f$/cm^2
σ_r	Radial unit stress	[illegible]/in.^{2}	kg$_f$/cm^2
σ_t	Tangential unit stress	[illegible]$_f$/in.^{2}	kg$_f$/cm^2
ϕ	Bending angle	rad	rad

	Term	[illegible] Units	[illegible] Units
		[illegible]	kg [illegible]
	[illegible] load support [illegible] [illegible] radius	[illegible]	kg-[illegible]
	[illegible] edge and/or load [illegible] [illegible]	[illegible]	kg-[illegible]
	[illegible] of integration [illegible] [illegible] entire [illegible] plate [illegible] by circles of [illegible] [illegible] plate supported [illegible]		[illegible]
	[illegible] of integration [illegible] [illegible] plate, [illegible] [illegible] bounded by circles [illegible] plate [illegible] radius		[illegible]
	[illegible]		[illegible]
	Modulus of elasticity		[illegible]
	[illegible] plate thickness	[illegible]	[illegible]
	Second moment of area per [illegible] length, symbolically $h^3/12$		[illegible]
	Maximum deflection [illegible]		[illegible]
	Maximum bending moment [illegible]		[illegible]
	Maximum bending [illegible] [illegible]	[illegible]	[illegible]

BENDING OF CIRCULAR PLATES UNDER A VARIABLE SYMMETRICAL LOAD

by

James C. Heap

ABSTRACT

The basic equations of deflection, slope, and moment for a thin, flat, circular plate, under a symmetrical variable load, for a constant force divided by the square of the radial distance, have been developed. Six cases have been derived. The first four cases cover the variable load acting over the entire plate, viz., (1) fixed, supported, outer edge and fixed, inner edge, (2) simply supported, outer edge and free, inner edge, (3) simply supported, outer edge and fixed, inner edge, (4) fixed, supported, outer edge and free, inner edge; and the final two cases are for a solid plate having the acting variable load bounded by circles of an inner radius and the outer support-load radius; i.e., (5) fixed, supported, outer edge and (6) simply supported, outer edge.

INTRODUCTION

In the design and development of the experimental apparatus for the Argonne National Laboratory Zero Gradient Synchrotron, the deflection, slope, and moment equations for a thin, flat, circular plate under a variable, symmetrical, electrical force had to be derived to establish structural integrity. The effect of a variable electrical force imposed on the thin, flat, circular plate is not presented in the commonly used references. It was therefore necessary to derive the required equations.

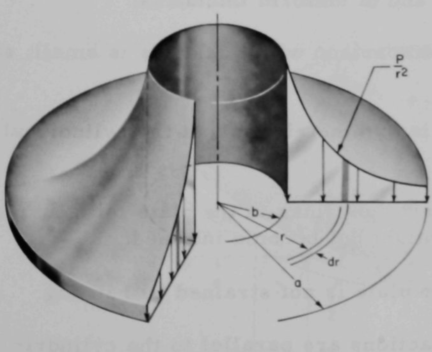


Fig. 1. Variable Symmetrical Load Distribution on Circular Plate; Schematic Diagram

This paper presents a treatment for six cases of varying load distribution in which constant force, divided by the radial distance squared (schematically delineated in Fig. 1), acts on a thin, flat, circular plate. In the first four cases the variable load is assumed to act over the entire actual plate; in the last two cases, solid plates are considered with a variable load over the plate bounded by circles of inner radius and the outer plate-load radius. The six cases are: (1) outer edge supported and fixed, inner edge fixed; (2) outer edge simply supported, inner edge free; (3) outer edge simply supported, inner edge fixed; (4) outer edge supported and fixed, inner edge free; (5) outer edge supported and fixed, solid plate; and (6) outer edge simply supported, solid plate.

A computer program was developed for ascertaining deflections and moments. To simplify the determination of deflections, moments, and slopes when only one or two calculations are required, various dimensionless terms in the derived equations have been computed and presented in tabular form. The maximum deflection constants for the six cases are graphically depicted. Bending-moment diagrams for these six cases have been obtained for a set of parameters. The maximum deflection and bending-moment constants are presented in a table for rapid computations using prescribed conditions.

SYSTEM OF UNITS

In this presentation, the unit force-mass system is used since it provides a compromise between the absolute and gravitational systems, and is automatically a self-containing reference system. Reference 1 contains a comprehensive analysis of this system.

SUPPOSITIONS

1. The plate under consideration is assumed to be perfectly elastic, isotropic (modulus of elasticity and Poisson's ratio are the same in all directions), and homogeneous.

2. The plate initially is flat and of uniform thickness.

3. Maximum deflection in comparison with thickness is small, say no more than half the thickness.

4. Deformation of the plate is symmetrical about the cylindrical axis.

5. During deformation, the straight lines in the plate initially parallel to the cylindrical axis remain straight but become inclined.

6. The middle surface of the plate is not strained by bending.

7. All forces, loads, and reactions are parallel to the cylindrical axis.

8. Shear effect on bending is negligible, thickness limited to no more than one-quarter of the least radial dimension.

9. Structural damping effect is neglected.

10. Temperature is uniform throughout the plate, and thermal equilibrium exists between the plate, the surrounding medium, and the support.

THEORETICAL ASPECT

The ensuing theoretical compendium has been included with several thoughts in mind, viz., (1) it is an abbreviated version, (2) it relates all necessary formulas, and (3) it eliminates acquiring a reference if a quick review is desired. The derived bending moments, slope, and deflection equations are the ones ascribed to Grashof and Poisson (see references 2 and 3). Additional reading on the theory is contained in references 2 through 6.

The pertinent unit-strain equations, according to Hooke's law for plane stress and the geometric relations illustrated by Fig. 2, are

$$\epsilon_r = \frac{\sigma_r}{E} - \nu\frac{\sigma_t}{E} = y\frac{d\phi}{dr}; \quad \epsilon_t = \frac{\sigma_t}{E} - \nu\frac{\sigma_r}{E} = y\frac{\phi}{r}. \tag{1}$$

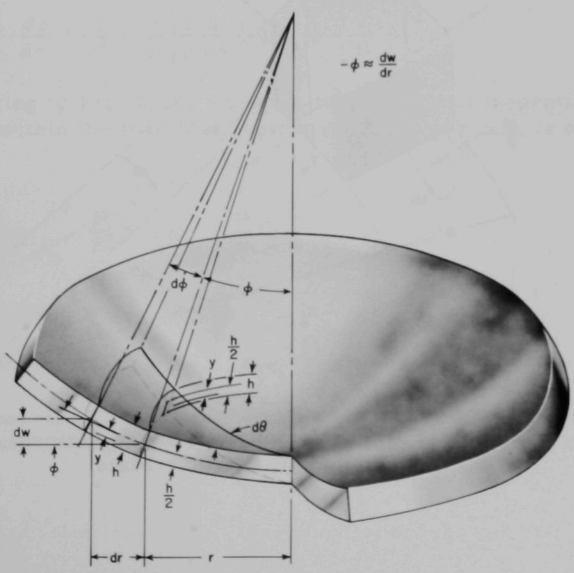


Fig. 2. Bending-Deflection Relationships for Element on Thin, Flat, Circular Plate

Solving for the radial and tangential unit stresses, one obtains

$$\sigma_r = \frac{Ey}{1-\nu^2}\left(\frac{d\phi}{dr} + \nu\frac{\phi}{r}\right);$$

$$\sigma_t = \frac{Ey}{1-\nu^2}\left(\frac{\phi}{r} + \nu\frac{d\phi}{dr}\right). \tag{2}$$

If it be assumed that unit stresses are proportional to the distances from the middle surface, then, through use of Figs. 2 and 3 and Eq. (2), the radial and tangential bending moments per unit length are

$$M_r = \int_{-h/2}^{h/2} \sigma_r\, y\, dA/\text{unit length} = D\, \frac{d\phi}{dr} + \nu\frac{\phi}{r}\ ;$$

$$M_t = \int_{-h/2}^{h/2} \sigma_t\, y\, dA/\text{unit length} = D\, \frac{\phi}{r} + \nu\frac{d\phi}{dr}\ , \tag{3}$$

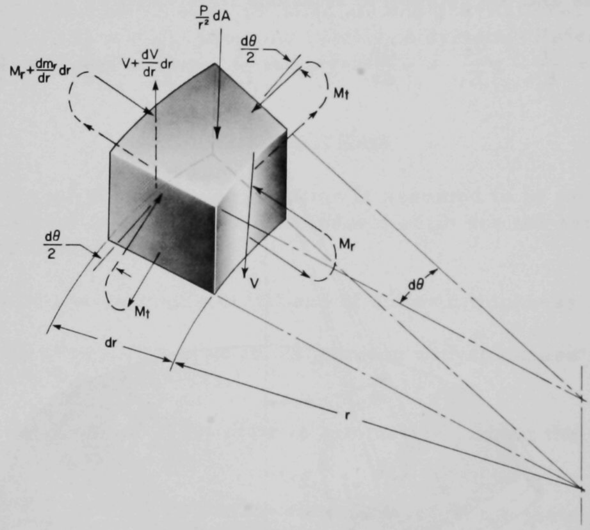


Fig. 3. Forces-Moments Acting on Element of Thin, Flat, Circular Plate

where

$$D = \frac{EI}{1 - \nu^2} = \frac{Eh^3}{12(1 - \nu^2)}.$$

Summation of the moments about the center tangential axis of the element shown in Fig. 3 gives

$$\Sigma M_t = 0 = \left(M_r + \frac{dM_r}{dr} dr\right)(r + dr)d\theta - M_r r \, d\theta - 2M_t \, dr \frac{d\theta}{2}$$

$$+ \left(V + \frac{dV}{dr} dr\right)(r + dr)\frac{dr}{2} d\theta + Vr \frac{dr}{2} d\theta, \tag{4}$$

where the trigonometric sine function has been assumed equal to the angle. Rearrangement of terms and neglect of higher-order derivatives of Eq. (4) yields

$$\frac{dM_r}{dr} + \frac{M_r - M_t}{r} = - V. \tag{5}$$

The equilibrium equation in terms of the bending angle and radius is now ascertained by taking the derivative of the first expression in Eq. (3), substituting this expression and the expressions of Eq. (3) into Eq. (5); thus,

$$\frac{d^2\phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} - \frac{\phi}{r^2} \equiv \frac{d}{dr}\left[\frac{1}{r} \frac{d}{dr} (r\phi)\right] = - \frac{V}{D}. \tag{6}$$

Referring to Fig. 1, the shearing force per unit tangential length at any radius within the load distribution region, $b \leq r \leq a$, is established as

$$V = \frac{1}{2\pi r} \int_b^r \frac{P}{r^2} 2\pi r \, dr = \frac{P}{r} \ln \frac{r}{b}, \tag{7}$$

and for the unloaded region, $0 \leq r \leq b$, is

$$V = 0. \tag{8}$$

The general equations for the load-distribution region, as a consequence of substituting Eq. (7) into Eq. (6) and then integrating, are

$$\frac{d}{dr}(r\phi) = -\frac{Pr}{D}\left[\frac{1}{2}(\ln r)^2 - \ln r\,(\ln b)\right] + C_1 r;$$

$$-\frac{dw}{dr} \approx \phi = -\frac{Pr}{4D}\left[\left(\ln\frac{r}{b}\right)^2 - \ln\frac{r}{b} - (\ln b)^2 + \frac{1}{2}\right] + \frac{C_1}{2}r + \frac{C_2}{r};$$

$$w = \frac{Pr^2}{8D}\left[\left(\ln\frac{r}{b} - 1\right)^2 - (\ln b)^2 + \frac{1}{2}\right] - \frac{C_1}{4}r^2 - C_2\ln r + C_3. \qquad (9)$$

The general equations for the unloaded region obtained by substituting Eq. (8) into Eq. (6) and integrating, are

$$\frac{d}{dr}(r\phi) = C_4 r;$$

$$-\frac{dw}{dr} \approx \phi = \frac{C_4}{2}r + \frac{C_5}{r};$$

$$w = -\frac{C_4}{4}r^2 - C_5\ln r + C_6. \qquad (10)$$

If Eq. (3) is used and the derivative taken of the second expressions of Eq. (9) and Eq. (10), the bending-moment equations for the loaded region become

$$M_r = -\frac{P}{4}(1+\nu)\left[\left(\ln\frac{r}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{r}{b} - (\ln b)^2 - \frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right)\right]$$
$$+ \frac{C_1 D}{2}(1+\nu) - \frac{C_2 D}{r^2}(1-\nu);$$

$$M_t = -\frac{P}{4}(1+\nu)\left[\left(\ln\frac{r}{b}\right)^2 - \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{r}{b} - (\ln b)^2 + \frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right)\right]$$
$$+ \frac{C_1 D}{2}(1+\nu) + \frac{C_2 D}{r^2}(1-\nu), \qquad (11)$$

and the bending-moment equations for the inner region become

$$M_r = D\left[\frac{C_4}{2}(1+\nu) - \frac{C_5}{r^2}(1-\nu)\right];$$

$$M_t = D\left[\frac{C_4}{2}(1+\nu) + \frac{C_5}{r^2}(1-\nu)\right]. \qquad (12)$$

The six cases presented in tabular form in the following pages were derived by using the appropriate equations that fulfill the continuity conditions and/or boundary conditions. The equations used in obtaining the integration constants were the last two expressions of Eqs. (9) and (10), plus the first expression of Eqs. (11) and (12). The continuity conditions and/or boundary conditions for each case are shown in the upper right corner of the tabulations. As an example, consider Case III. The boundary conditions are

$$w = 0 \quad \text{when } r = a;$$

$$M_r = 0 \quad \text{when } r = a;$$

$$\frac{dw}{dr} = 0 \quad \text{when } r = b. \tag{13}$$

Hence, the three equations to be solved for the constants are

$$0 = \frac{Pa^2}{8D}\left[\left(\ln\frac{a}{b} - 1\right)^2 - (\ln b)^2 + \frac{1}{2}\right] - \frac{C_1}{4}a^2 - C_2\ln a + C_3;$$

$$0 = -P(1+\nu)\left[\left(\ln\frac{a}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{a}{b} - (\ln b)^2 - \frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right)\right]$$

$$+ \frac{C_1 D}{2}(1+\nu) - \frac{C_2 D}{a^2}(1-\nu);$$

$$0 = -\frac{Pb}{4D}\left[-(\ln b)^2 + \frac{1}{2}\right] + \frac{C_1}{2}b + \frac{C_2}{b}, \tag{14}$$

where the second and third expressions of Eq. (9) and the first expression of Eq. (11) were used.

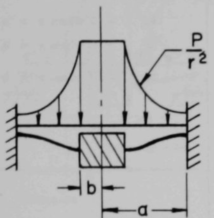
To facilitate the moment, slope, and deflection computations, various terms in the derived formulas have been computed and are related in Table I.

TABLE I

COMPUTATION TERMS

$\frac{a}{b}$	$\frac{b^2}{a^2}$	$1-\frac{b^2}{a^2}$	$1+\frac{b^2}{a^2}$	$\frac{b^2}{a^2-b^2}$	$\frac{a^2}{a^2-b^2}$	$\ln\frac{a}{b}$	$\left(\ln\frac{a}{b}\right)^2$	$\ln\frac{a}{b}-1$	$\left(\ln\frac{a}{b}-1\right)^2$	$\frac{b^2}{a^2}\left(\ln\frac{a}{b}\right)$	$\frac{b^2}{a^2}\left(\ln\frac{a}{b}\right)^2$	$\left(1+\frac{b^2}{a^2}\right)\ln\frac{a}{b}$	$\frac{b^2}{a^2-b^2}\left(\ln\frac{a}{b}\right)$	$\frac{b^2}{a^2-b^2}\left(\ln\frac{a}{b}\right)^2$	$\frac{a^2}{a^2-b^2}\left(\ln\frac{a}{b}\right)$	$\frac{a^2}{a^2-b^2}\left(\ln\frac{a}{b}\right)^2$
1.0	1.00000	0.00000	2.00000	∞	∞	0.00000	0.00000	-1.00000	1.00000	0.00000	0.00000	0.00000	0.50000	0.00000	0.50000	0.00000
1.1	0.82645	0.17355	1.82645	4.76190	5.76190	0.09531	0.00908	-0.90469	0.81846	0.07877	0.00750	0.17408	0.45386	0.04324	0.54917	0.05232
1.2	0.69444	0.30556	1.69444	2.27273	3.27273	0.18232	0.03324	-0.81768	0.66860	0.12661	0.02308	0.30893	0.41436	0.07555	0.59668	0.10879
1.3	0.59172	0.40828	1.59172	1.44928	2.44928	0.26236	0.06883	-0.73764	0.54411	0.15524	0.04073	0.41760	0.38023	0.09975	0.64259	0.16858
1.4	0.51020	0.48980	1.51020	1.04167	2.04167	0.33647	0.11321	-0.66353	0.44027	0.17167	0.05776	0.50814	0.35049	0.11793	0.68696	0.23114
1.5	0.44444	0.55556	1.44444	0.80000	1.80000	0.40547	0.16441	-0.59453	0.35347	0.18021	0.07307	0.58568	0.32438	0.13153	0.72985	0.29594
1.6	0.39063	0.60937	1.39063	0.64103	1.64103	0.47000	0.22090	-0.53000	0.28090	0.18360	0.08629	0.65360	0.30128	0.14160	0.77128	0.36250
1.7	0.34602	0.65398	1.34602	0.52910	1.52910	0.53063	0.28157	-0.46937	0.22031	0.18361	0.09743	0.71424	0.28076	0.14898	0.81139	0.43055
1.8	0.30864	0.69136	1.30864	0.44643	1.44643	0.58779	0.34550	-0.41221	0.16992	0.18142	0.10664	0.76921	0.26241	0.15424	0.85020	0.49974
1.9	0.27701	0.72299	1.27701	0.38314	1.38314	0.64185	0.41197	-0.35815	0.12827	0.17780	0.11412	0.81965	0.24592	0.15784	0.88777	0.56981
2.0	0.25000	0.75000	1.25000	0.33333	1.33333	0.69315	0.48046	-0.30685	0.09416	0.17329	0.12012	0.86644	0.23105	0.16015	0.92420	0.64061
2.1	0.22676	0.77324	1.22676	0.29326	1.29326	0.74194	0.55047	-0.25806	0.06659	0.16824	0.12482	0.91018	0.21758	0.16143	0.95952	0.71190
2.2	0.20661	0.79339	1.20661	0.26042	1.26042	0.78846	0.62167	-0.21154	0.04475	0.16290	0.12844	0.95136	0.20533	0.16189	0.99379	0.78357
2.3	0.18904	0.81096	1.18904	0.23310	1.23310	0.83291	0.69374	-0.16709	0.02792	0.15745	0.13114	0.99036	0.19415	0.16171	1.02706	0.85545
2.4	0.17361	0.82639	1.17361	0.21008	1.21008	0.87547	0.76645	-0.12453	0.01551	0.15199	0.13306	1.02746	0.18392	0.16102	1.05939	0.92747
2.5	0.16000	0.84000	1.16000	0.19048	1.19048	0.91629	0.83959	-0.08371	0.00701	0.14661	0.13433	1.06290	0.17453	0.15992	1.09082	0.99952
2.6	0.14793	0.85207	1.14793	0.17361	1.17361	0.95551	0.91300	-0.04449	0.00198	0.14135	0.13506	1.09686	0.16589	0.15851	1.12140	1.07151
2.7	0.13717	0.86283	1.13717	0.15898	1.15898	0.99325	0.98655	-0.00675	0.00005	0.13624	0.13533	1.12949	0.15791	0.15684	1.15116	1.14339
2.8	0.12755	0.87245	1.12755	0.14620	1.14620	1.02962	1.06012	0.02962	0.00088	0.13133	0.13522	1.16095	0.15053	0.15499	1.18015	1.21511
2.9	0.11891	0.88109	1.11891	0.13495	1.13495	1.06471	1.13361	0.06471	0.00419	0.12660	0.13480	1.19131	0.14369	0.15298	1.20839	1.28659
3.0	0.11111	0.88889	1.11111	0.12500	1.12500	1.09861	1.20694	0.09861	0.00972	0.12207	0.13410	1.22068	0.13733	0.15087	1.23594	1.35781
3.1	0.10406	0.89594	1.10406	0.11614	1.11614	1.13140	1.28007	0.13140	0.01727	0.11773	0.13320	1.24913	0.13141	0.14867	1.26280	1.42874
3.2	0.09766	0.90234	1.09766	0.10823	1.10823	1.16315	1.35292	0.16315	0.02662	0.11359	0.13213	1.27674	0.12588	0.14642	1.28904	1.49935
3.3	0.09183	0.90817	1.09183	0.10111	1.10111	1.19392	1.42544	0.19392	0.03760	0.10964	0.13090	1.30356	0.12072	0.14413	1.31464	1.56957
3.4	0.08651	0.91349	1.08651	0.09470	1.09470	1.22378	1.49764	0.22378	0.05008	0.10587	0.12956	1.32965	0.11589	0.14182	1.33967	1.63947
3.5	0.08163	0.91837	1.08163	0.08889	1.08889	1.25276	1.56941	0.25276	0.06389	0.10226	0.12811	1.35502	0.11136	0.13950	1.36412	1.70891
3.6	0.07716	0.92284	1.07716	0.08361	1.08361	1.28093	1.64078	0.28093	0.07892	0.09884	0.12660	1.37977	0.10710	0.13719	1.38803	1.77797
3.7	0.07305	0.92695	1.07305	0.07880	1.07880	1.30833	1.71173	0.30833	0.09507	0.09557	0.12500	1.40390	0.10310	0.13489	1.41143	1.84661
3.8	0.06925	0.93075	1.06925	0.07440	1.07440	1.33500	1.78223	0.33500	0.11223	0.09245	0.12342	1.42745	0.09933	0.13261	1.43432	1.91483
3.9	0.06575	0.93425	1.06575	0.07037	1.07037	1.36098	1.85227	0.36098	0.13031	0.08948	0.12179	1.45046	0.09578	0.13035	1.45675	1.98261
4.0	0.06250	0.93750	1.06250	0.06667	1.06667	1.38629	1.92180	0.38629	0.14922	0.08664	0.12011	1.47293	0.09242	0.12812	1.47871	2.04993

CASE I



Description

Outer edge supported and fixed.

Inner edge fixed.

Variable load over entire actual plate.

Boundary Conditions

$w = 0$ when $r = a$

$\frac{dw}{dr} = 0$ when $r = a$

$\frac{dw}{dr} = 0$ when $r = b$

Moments

$$M_r = \frac{P}{4}(1+\nu)\left\{\frac{a^2}{a^2-b^2}\left[\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b}\right]\left[1+\left(\frac{1-\nu}{1+\nu}\right)\frac{b^2}{r^2}\right] - \left(\ln\frac{r}{b}\right)^2 - \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{r}{b} + \frac{1}{1+\nu}\right\}$$

$$M_{rb} = \frac{P}{4}\left\{1+\frac{2a^2}{a^2-b^2}\left[\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b}\right]\right\} \text{ (Maximum when } a/b > e\text{)}$$

$$M_{ra} = \frac{P}{4}\left\{1+\frac{2a^2}{a^2-b^2}\left[\frac{b^2}{a^2}\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b}\right]\right\} \text{ (Maximum when } a/b < e\text{)}$$

$$M_t = \frac{P}{4}(1+\nu)\left\{\frac{a^2}{a^2-b^2}\left[\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b}\right]\left[1-\left(\frac{1-\nu}{1+\nu}\right)\frac{b^2}{r^2}\right] - \left(\ln\frac{r}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{r}{b} + \frac{\nu}{1+\nu}\right\}$$

$$M_{tb} = \nu M_{rb}; \quad M_{ta} = \nu M_{ra}$$

Slope

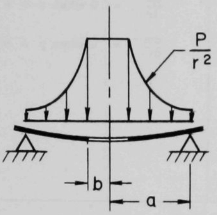
$$\frac{dw}{dr} = -\frac{Pr}{4D}\left\{-\left(\ln\frac{r}{b}\right)^2 + \ln\frac{r}{b} + \left(1-\frac{b^2}{r^2}\right)\left(\frac{a^2}{a^2-b^2}\right)\left[\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b}\right]\right\}$$

Deflection

$$w = \frac{Pa^2}{8D}\left\{\frac{r^2}{a^2}\left(\ln\frac{r}{b}-1\right)^2 - \left(\ln\frac{a}{b}-1\right)^2 + \left[\left(1-\frac{r^2}{a^2}\right) - 2\frac{b^2}{a^2}\left(\ln\frac{a}{r}\right)\right]\left[\frac{a^2}{a^2-b^2}\right]\left[\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b}\right]\right\}$$

$$w_{max} = \frac{Pa^2}{8D}\left\{\frac{b^2}{a^2} + \left(\ln\frac{a}{b}-1\right)\left[1-\frac{2b^2}{a^2-b^2}\left(\ln\frac{a}{b}\right)^2\right]\right\}$$

CASE II



Description

Outer edge simply supported.

Inner edge free.

Variable load over entire actual plate.

Boundary Conditions

$w = 0$ when $r = a$

$M_r = 0$ when $r = a$

$M_r = 0$ when $r = b$

Moments

$$M_r = \frac{P}{4}(1+\nu)\left\{-\left(\ln\frac{r}{b}\right)^2 + \left(1-\frac{b^2}{r^2}\right)\left(\frac{a^2}{a^2-b^2}\right)\left(\ln\frac{a}{b}\right)^2 - \left(\frac{1-\nu}{1+\nu}\right)\left[\ln\frac{r}{b} - \left(1-\frac{b^2}{r^2}\right)\left(\frac{a^2}{a^2-b^2}\right)\ln\frac{a}{b}\right]\right\}$$

$$M_t = \frac{P}{4}(1+\nu)\left\{-\left(\ln\frac{r}{b}\right)^2 + \left(1+\frac{b^2}{r^2}\right)\left(\frac{a^2}{a^2-b^2}\right)\left(\ln\frac{a}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\left[\ln\frac{r}{b} + \left(1+\frac{b^2}{r^2}\right)\left(\frac{a^2}{a^2-b^2}\right)\ln\frac{a}{b}\right] - \left(\frac{1-\nu}{1+\nu}\right)\right\}$$

$$M_{max} = M_{tb} = \frac{P}{4}(1+\nu)\left\{\frac{2a^2}{a^2-b^2}\left[\left(\ln\frac{a}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{a}{b}\right] - \left(\frac{1-\nu}{1+\nu}\right)\right\}$$

$$M_{ta} = \frac{P}{4}(1+\nu)\left\{\frac{2a^2}{a^2-b^2}\left[\frac{b^2}{a^2}\left(\ln\frac{a}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{a}{b}\right] - \left(\frac{1-\nu}{1+\nu}\right)\right\}$$

Slope

$$\frac{dw}{dr} = -\frac{Pr}{4D}\left\{-\left(\ln\frac{r}{b}\right)^2 + \left(\ln\frac{r}{b}\right) - \frac{1}{1+\nu} + \left[\left(\frac{1-\nu}{1+\nu}\right) + \frac{b^2}{r^2}\right]\left[\frac{a^2}{a^2-b^2}\right]\left[\left(\frac{1+\nu}{1-\nu}\right)\left(\ln\frac{a}{b}\right)^2 + \ln\frac{a}{b}\right]\right\}$$

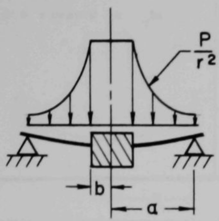
Deflection

$$w = \frac{Pa^2}{8D}\left\{\frac{r^2}{a^2}\left(\ln\frac{r}{b} - 1\right)^2 - \left(\ln\frac{a}{b} - 1\right)^2 + \left(1-\frac{r^2}{a^2}\right)\left(\frac{a^2}{a^2-b^2}\right)\left[\left(\ln\frac{a}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{a}{b}\right]\right.$$

$$\left. + \frac{2b^2}{a^2-b^2}\left[\left(\frac{1+\nu}{1-\nu}\right)\left(\ln\frac{a}{b}\right)^2 + \ln\frac{a}{b}\right]\left(\ln\frac{a}{r}\right) - \frac{1}{1+\nu}\left(1-\frac{r^2}{a^2}\right)\right\}$$

$$w_{max} = \frac{Pa^2}{8D}\left\{-\left(\frac{2+\nu}{1+\nu}\right)\left(1-\frac{b^2}{a^2}\right) + \left(\frac{3+\nu}{1+\nu}\right)\ln\frac{a}{b} + \frac{2b^2}{a^2-b^2}\left(\ln\frac{a}{b}\right)\left[\left(\frac{1+\nu}{1-\nu}\right)\left(\ln\frac{a}{b}\right)^2 + \ln\frac{a}{b}\right]\right\}$$

CASE III



Description

Outer edge simply supported.

Inner edge fixed.

Variable load over entire actual plate.

Boundary Conditions

$w = 0$ when $r = a$

$M_r = 0$ when $r = a$

$\frac{dw}{dr} = 0$ when $r = b$

Moments

$$M_r = \frac{P}{4}(1+\nu)\left\{-\left(\ln\frac{r}{b}\right)^2 - \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{r}{b} + \frac{1}{1+\nu} + \left[\frac{\frac{1+\nu}{1-\nu}+\frac{b^2}{r^2}}{\frac{1+\nu}{1-\nu}+\frac{b^2}{a^2}}\right]\left[\left(\ln\frac{a}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{a}{b} - \frac{1}{1+\nu}\right]\right\}$$

$$M_{rb} = \frac{P}{2}\left[\frac{1}{\frac{1+\nu}{1-\nu}+\frac{b^2}{a^2}}\right]\left[\left(\frac{1+\nu}{1-\nu}\right)\left(\ln\frac{a}{b}\right)^2 + \ln\frac{a}{b} - \frac{1}{2}\left(1-\frac{b^2}{a^2}\right)\right] \quad \text{(Maximum, } \nu = 0.3\text{)}$$

$$M_t = \frac{P}{4}(1+\nu)\left\{-\left(\ln\frac{r}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{r}{b} + \frac{\nu}{1+\nu} + \left[\frac{\frac{1+\nu}{1-\nu}-\frac{b^2}{r^2}}{\frac{1+\nu}{1-\nu}+\frac{b^2}{a^2}}\right]\left[\left(\ln\frac{a}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{a}{b} - \frac{1}{1+\nu}\right]\right\}$$

$$M_{tb} = \nu M_{rb}$$

$$M_{ta} = \frac{P}{2}(1+\nu)\left[\frac{-1}{\frac{1+\nu}{1-\nu}+\frac{b^2}{a^2}}\right]\left[\frac{b^2}{a^2}\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b} + \frac{1}{2}\left(1-\frac{b^2}{a^2}\right)\right]$$

Slope

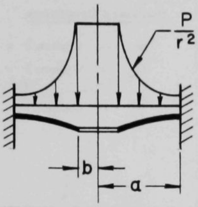
$$\frac{dw}{dr} = -\frac{Pr}{4D}\left\{-\left(\ln\frac{r}{b}\right)^2 + \ln\frac{r}{b} + \left[\frac{1-\frac{b^2}{r^2}}{\frac{1+\nu}{1-\nu}+\frac{b^2}{a^2}}\right]\left[\left(\frac{1+\nu}{1-\nu}\right)\left(\ln\frac{a}{b}\right)^2 + \ln\frac{a}{b} - \frac{1}{1-\nu}\right]\right\}$$

Deflection

$$w = \frac{Pa^2}{8D}\left\{\frac{r^2}{a^2}\left(\ln\frac{r}{b}-1\right)^2 - \left(\ln\frac{a}{b}-1\right)^2 + \left[\frac{\left(1-\frac{r^2}{a^2}\right) - 2\frac{b^2}{a^2}\left(\ln\frac{a}{r}\right)}{\frac{1+\nu}{1-\nu}+\frac{b^2}{a^2}}\right]\left[\left(\frac{1+\nu}{1-\nu}\right)\left(\ln\frac{a}{b}\right)^2 + \ln\frac{a}{b} - \frac{1}{1-\nu}\right]\right\}$$

$$w_{max} = \frac{Pa^2}{8D}\left\{\frac{b^2}{a^2} - \left(\ln\frac{a}{b}-1\right)^2 + \left[\frac{\left(1-\frac{b^2}{a^2}\right) - 2\frac{b^2}{a^2}\left(\ln\frac{a}{b}\right)}{\frac{1+\nu}{1-\nu}+\frac{b^2}{a^2}}\right]\left[\left(\frac{1+\nu}{1-\nu}\right)\left(\ln\frac{a}{b}\right)^2 + \ln\frac{a}{b} - \frac{1}{1-\nu}\right]\right\}$$

CASE IV



Description

Outer edge supported and fixed.

Inner edge free.

Variable load over entire actual plate.

Boundary Conditions

$w = 0$ when $r = a$

$\frac{dw}{dr} = 0$ when $r = a$

$M_r = 0$ when $r = b$

Moments

$$M_r = \frac{P}{4}(1+\nu)\left\{-\left(\ln\frac{r}{b}\right)^2 - \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{r}{b} + \left[\frac{1-\frac{b^2}{r^2}}{1+\left(\frac{1+\nu}{1-\nu}\right)\frac{b^2}{a^2}}\right]\left[\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b} + \frac{1}{1+\nu}\right]\right\}$$

$$M_{ra} = \frac{P}{2}\left[\frac{-1}{1+\left(\frac{1+\nu}{1-\nu}\right)\frac{b^2}{a^2}}\right]\left[\left(\frac{1+\nu}{1-\nu}\right)\left(\frac{b^2}{a^2}\right)\left(\ln\frac{a}{b}\right)^2 + \ln\frac{a}{b} - \frac{1}{2}\left(1-\frac{b^2}{a^2}\right)\right]$$ (Maximum when $a/b < 3.56$; $\nu = 0.3$

$$M_t = \frac{P}{4}(1+\nu)\left\{-\left(\ln\frac{r}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{r}{b} - \frac{1-\nu}{1+\nu} + \left[\frac{1+\frac{b^2}{r^2}}{1+\left(\frac{1+\nu}{1-\nu}\right)\frac{b^2}{a^2}}\right]\left[\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b} + \frac{1}{1+\nu}\right]\right\}$$

$$M_{tb} = \frac{P}{2}(1+\nu)\left[\frac{1}{1+\left(\frac{1+\nu}{1-\nu}\right)\frac{b^2}{a^2}}\right]\left[\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b} + \frac{1}{2}\left(1-\frac{b^2}{a^2}\right)\right]$$ (Maximum when $a/b > 3.56$; $\nu = 0.3$

$$M_{ta} = \nu M_{ra}$$

Slope

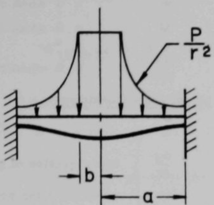
$$\frac{dw}{dr} = -\frac{Pr}{4D}\left\{-\left(\ln\frac{r}{b}\right)^2 + \ln\frac{r}{b} - \frac{1}{1+\nu} + \left[\frac{\left(\frac{1-\nu}{1+\nu}\right)+\frac{b^2}{r^2}}{\left(\frac{1-\nu}{1+\nu}\right)+\frac{b^2}{a^2}}\right]\left[\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b} + \frac{1}{1+\nu}\right]\right\}$$

Deflection

$$w = \frac{Pa^2}{8D}\left\{\frac{r^2}{a^2}\left(\ln\frac{r}{b}-1\right)^2 - \left(\ln\frac{a}{b}-1\right)^2 - \frac{1}{1+\nu}\left(1-\frac{r^2}{a^2}\right) + \left[\frac{\left(\frac{1-\nu}{1+\nu}\right)\left(1-\frac{r^2}{a^2}\right) + 2\frac{b^2}{a^2}\left(\ln\frac{a}{r}\right)}{\left(\frac{1-\nu}{1+\nu}\right)+\frac{b^2}{a^2}}\right]\left[\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b} + \frac{1}{1+\nu}\right]\right\}$$

$$w_{max} = \frac{Pa^2}{8D}\left\{\frac{b^2}{a^2} - \left(\ln\frac{a}{b}-1\right)^2 - \frac{1}{1+\nu}\left(1-\frac{b^2}{a^2}\right) + \left[\frac{\left(\frac{1-\nu}{1+\nu}\right)\left(1-\frac{b^2}{a^2}\right) + 2\frac{b^2}{a^2}\ln\frac{a}{b}}{\left(\frac{1-\nu}{1+\nu}\right)+\frac{b^2}{a^2}}\right]\left[\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b} + \frac{1}{1+\nu}\right]\right\}$$

CASE V



Description

Outer edge supported and fixed.

Solid Plate.

Variable load over plate bounded by circles of inner radius and outer plate radius.

Boundary Conditions

$w = 0$ when $r = a$

$\frac{dw}{dr} = 0$ when $r = a$

$\frac{dw}{dr} = 0$ when $r = 0$

Continuity Conditions

w, $\frac{dw}{dr}$, M_r when $r = b$ inner portion of plate same as outer portion

Inner portion of plate, $r = 0$ to $r = b$

Moments

$$M_r = M_t = M_{rb} = M_{tb} = \frac{P}{4}(1+\nu)\left[\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b} + \frac{1}{2}\left(1 - \frac{b^2}{a^2}\right)\right] \text{(Maximum when } a/b \geq 6.55;\ \nu = 0.3)$$

Slope

$$\frac{dw}{dr} = -\frac{Pr}{4D}\left[\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b} + \frac{1}{2}\left(1 - \frac{b^2}{a^2}\right)\right]$$

Deflection

$$w = \frac{Pa^2}{8D}\left\{-\left(\frac{r}{a}\right)^2\left[\left(\ln\frac{a}{b}\right)^2 - \ln\frac{a}{b} + \frac{1}{2}\left(1 - \frac{b^2}{a^2}\right)\right] + \left(1 + \frac{b^2}{a^2}\right)\ln\frac{a}{b} - \left(1 - \frac{b^2}{a^2}\right)\right\}$$

$$w_{max} = \frac{Pa^2}{8D}\left[\left(1 + \frac{b^2}{a^2}\right)\ln\frac{a}{b} - \left(1 - \frac{b^2}{a^2}\right)\right]$$

Outer portion of plate, $r = b$ to $r = a$

Moments

$$M_r = \frac{P}{4}(1+\nu)\left\{\left(\ln\frac{a}{b}\right)^2 - \left(\ln\frac{r}{b}\right)^2 - \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{r}{b} - \ln\frac{a}{b} + \frac{1}{1+\nu} - \frac{1}{2}\left[\frac{b^2}{a^2} + \left(\frac{1-\nu}{1+\nu}\right)\frac{b^2}{r^2}\right]\right\}$$

$$M_{ra} = \frac{P}{4}\left[-2\ln\frac{a}{b} + \left(1 - \frac{b^2}{a^2}\right)\right] \text{(Maximum when } a/b < 6.55;\ \nu = 0.3)$$

$$M_t = \frac{P}{4}(1+\nu)\left\{\left(\ln\frac{a}{b}\right)^2 - \left(\ln\frac{r}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{r}{b} - \ln\frac{a}{b} + \frac{\nu}{1+\nu} - \frac{1}{2}\left[\frac{b^2}{a^2} - \left(\frac{1-\nu}{1+\nu}\right)\frac{b^2}{r^2}\right]\right\}$$

$$M_{ta} = \nu M_{ra}$$

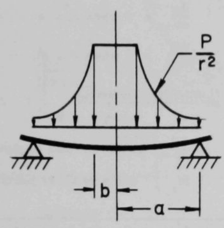
Slope

$$\frac{dw}{dr} = -\frac{Pr}{4D}\left[\left(\ln\frac{a}{b}\right)^2 - \left(\ln\frac{r}{b}\right)^2 - \ln\frac{a}{r} - \frac{1}{2}\frac{b^2}{a^2}\left(1 - \frac{a^2}{r^2}\right)\right]$$

Deflection

$$w = \frac{Pa^2}{8D}\left\{\frac{r^2}{a^2}\left[\ln\frac{r}{b} - 1\right]^2 - \frac{r^2}{a^2}\left(\ln\frac{a}{b}\right)^2 + \left(1 + \frac{r^2}{a^2}\right)\ln\frac{a}{b} - 1 + \frac{b^2}{a^2}\ln\frac{a}{r} - \frac{1}{2}\frac{b^2}{a^2}\left(1 - \frac{r^2}{a^2}\right)\right\}$$

CASE VI



Description

Outer edge simply supported.

Solid Plate.

Variable load over plate bounded by circles of inner radius and outer plate radius.

Boundary Conditions

$w = 0$ when $r = a$

$M_r = 0$ when $r = a$

$\frac{dw}{dr} = 0$ when $r = 0$

Continuity Conditions

w, $\frac{dw}{dr}$, M_r when $r = b$ inner portion of plate same as outer portion

Inner portion of plate, $r = 0$ to $r = b$

Moments

$$M_{max} = M_r = M_t = M_{rb} = M_{tb} = \frac{P}{4}(1+\nu)\left[\left(\ln\frac{a}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{a}{b} - \frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right)\left(1-\frac{b^2}{a^2}\right)\right]$$

Slope

$$\frac{dw}{dr} = -\frac{Pr}{4D}\left[\left(\ln\frac{a}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{a}{b} - \frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right)\left(1-\frac{b^2}{a^2}\right)\right]$$

Deflection

$$w = \frac{Pa^2}{8D}\left\{-\left(\frac{r}{a}\right)^2\left[\left(\ln\frac{a}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{a}{b} - \frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right)\left(1-\frac{b^2}{a^2}\right)\right] + \left(\frac{3+\nu}{1+\nu} + \frac{b^2}{a^2}\right)\ln\frac{a}{b} - \left(\frac{2+\nu}{1+\nu}\right)\left(1-\frac{b^2}{a^2}\right)\right\}$$

$$w_{max} = \frac{Pa^2}{8D}\left[\left(\frac{3+\nu}{1+\nu} + \frac{b^2}{a^2}\right)\ln\frac{a}{b} - \left(\frac{2+\nu}{1+\nu}\right)\left(1-\frac{b^2}{a^2}\right)\right]$$

Outer portion of plate, $r = b$ to $r = a$

Moments

$$M_r = \frac{P}{4}(1+\nu)\left[\left(\ln\frac{a}{b}\right)^2 - \left(\ln\frac{r}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{a}{r} - \frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right)\left(\frac{b^2}{r^2}\right)\left(1-\frac{r^2}{a^2}\right)\right]$$

$$M_t = \frac{P}{4}(1+\nu)\left\{\left(\ln\frac{a}{b}\right)^2 - \left(\ln\frac{r}{b}\right)^2 + \left(\frac{1-\nu}{1+\nu}\right)\left(\ln\frac{a}{b} + \ln\frac{r}{b}\right) - \left(\frac{1-\nu}{1+\nu}\right)\left[1 - \frac{1}{2}\left(\frac{b^2}{r^2}\right)\left(1+\frac{r^2}{a^2}\right)\right]\right\}$$

$$M_{ta} = \frac{P}{4}(1-\nu)\left[2\ln\frac{a}{b} - \left(1-\frac{b^2}{a^2}\right)\right]$$

Slope

$$\frac{dw}{dr} = -\frac{Pr}{4D}\left[-\left(\ln\frac{r}{b}\right)^2 + \left(\ln\frac{a}{b}\right)^2 + \ln\frac{r}{b} + \left(\frac{1-\nu}{1+\nu}\right)\ln\frac{a}{b} - \frac{1}{2}\left(1-\frac{b^2}{r^2}\right) - \frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right)\left(1-\frac{b^2}{a^2}\right)\right]$$

Deflection

$$w = \frac{Pa^2}{8D}\left\{\frac{r^2}{a^2}\left[\left(\ln\frac{r}{b}-1\right)^2 - \left(\ln\frac{a}{b}-1\right)^2\right] + \frac{b^2}{a^2}\left(\ln\frac{a}{r}\right) + \left(1-\frac{r^2}{a^2}\right)\left[\left(\frac{3+\nu}{1+\nu}\right)\ln\frac{a}{b} - \frac{3}{2} - \frac{1}{2}\left(\frac{1-\nu}{1+\nu}\right)\left(1-\frac{b^2}{a^2}\right)\right]\right\}$$

DESIGN CONSIDERATIONS

Normally, the maximum deflection and the maximum bending moment are the major design criteria. For these six cases, the maximum deflection can be represented by a formula of the type

$$w_{max} = k_d\, Pa^2/Eh^3 . \tag{15}$$

The maximum bending moment in all six cases can be expressed by the form

$$M_{max} = k_m P, \tag{16}$$

where "maximum" signifies magnitude only, or maximum absolute value.

Figure 4 depicts the maximum deflection constant for the six derived cases for ratios of the outer plate support and load-distribution radius, to the inner plate radius and/or load-distribution radius, from one through four. The determination of these deflection constants is based on a Poisson's ratio of 0.3. Numerical values of the deflection constant, calculated for several values of the ratio a/b and ν = 0.3, are tabulated in Table II.

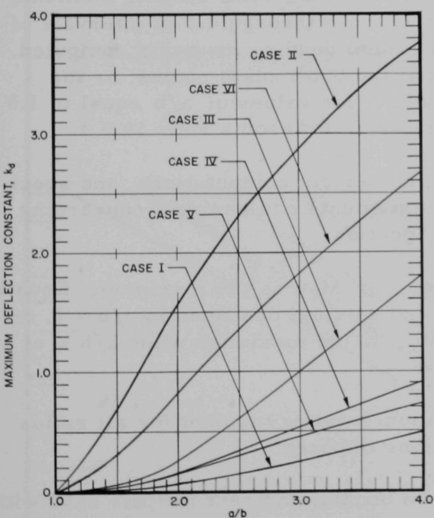


Fig. 4

Maximum Deflection Constant Versus Ratio of Outer Plate Support and Load Distribution Radius to Inner Plate and/or Load Distribution Radius for ν = 0.3

Since the bending moment must be an absolute maximum in determining the maximum stress, location and magnitude of the bending moment are a prerequisite. Because of the complexity of the moment equations, and because Poisson's ratio depends upon the material and related parameters, only the absolute maximum bending moment of Case VI could be verified by the customary mathematical procedures. Theoretically, Poisson's ratio can have a value from zero to 0.5; e.g., Poisson's ratio is approximately zero for cork, and nearly 0.5 for materials like paraffin and rubber.

TABLE II

MAXIMUM DEFLECTION AND MOMENT CONSTANTS WHERE ν = 0.3

a/b	Case I		Case II		Case III		Case IV		Case V		Case VI	
	k_d	k_m	k_d	k_m	k_d	k_m	k_d	k_m	k_d	k_m	k_d	k_m
1.5	0.009	-0.049	0.693	0.273	0.041	0.094	0.025	-0.072	0.041	-0.064	0.309	0.076
2.0	0.057	-0.132	1.591	0.565	0.254	0.287	0.148	-0.185	0.159	-0.159	0.827	0.212
2.5	0.141	-0.215	2.326	0.857	0.605	0.510	0.344	-0.287	0.304	-0.248	1.346	0.360
3.0	0.246	0.311	2.912	1.140	0.993	0.736	0.538	-0.374	0.453	-0.327	1.827	0.507
3.5	0.360	0.422	3.390	1.413	1.401	0.956	0.731	-0.448	0.596	-0.397	2.263	0.649
4.0	0.478	0.536	3.790	1.675	1.798	1.169	0.907	0.585	0.731	-0.459	2.658	0.785

To obtain a better insight into the bending moments for these six cases, Figs. 5 through 10 show the radial and tangential bending moments divided by the force constant, where a/b = 1.5 through 4.0 in intervals of 0.5, and ν = 0.3. Table II lists the maximum bending moments computed. These maximum moments are located at the outer plate radius, or the inner plate and/or load-distribution radius, for values of a/b equal to 1.5 through 4.0 in increments of 0.5, where again Poisson's ratio is 0.3.

From these moment diagrams, numerical computations, and specified conditions, the following general statements can be made concerning the maximum bending moment and its location:

Case I. From Fig. 5, either M_{ra} or M_{rb} is the maximum. Equating the absolute M_{ra} and M_{rb} equations and solving, one finds $\ln a/b = 1$, or $a/b = e = 2.71828 \ldots$. Therefore, M_{ra} is the maximum when $a/b < e$, and M_{rb} is the maximum when $a/b > e$.

Case II. M_{tb} yielded the maximum bending moment for all ratios of a/b = 1.5 through 4.0 for the conditions imposed.

Case III. M_{rb} had the maximum bending moment for this case with ν = 0.3.

Case IV. Here the maximum bending moment must be established according to specifications. With ν = 0.3, M_{ra} is the maximum when a/b is 3.56 or less, and M_{tb} is the maximum when a/b is greater than 3.56.

Case V. M_{ra} was the maximum calculated bending moment throughout the range covered. Transition a/b ratio is about 6.55, ν = 0.3.

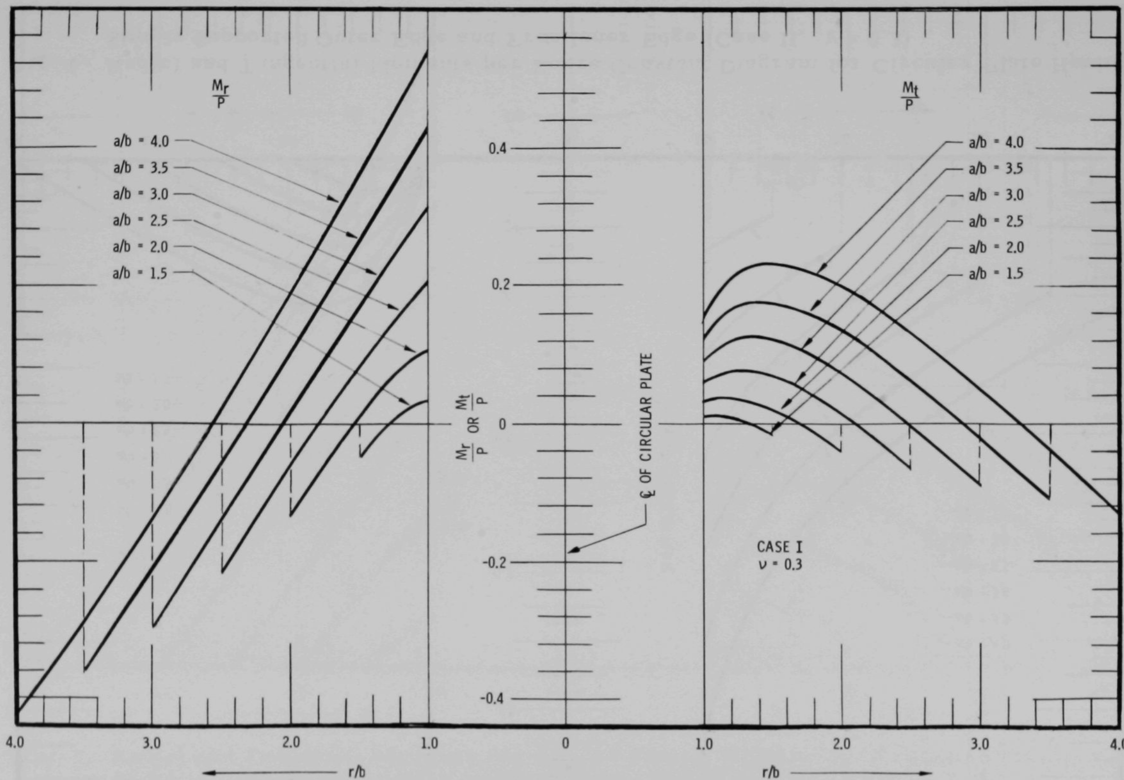


Fig. 5. Radial and Tangential Moments per Force Constant Diagram for Circular Plate Having Fixed Supported Outer Edge and Fixed Inner Edge (Case I, $\nu = 0.3$)

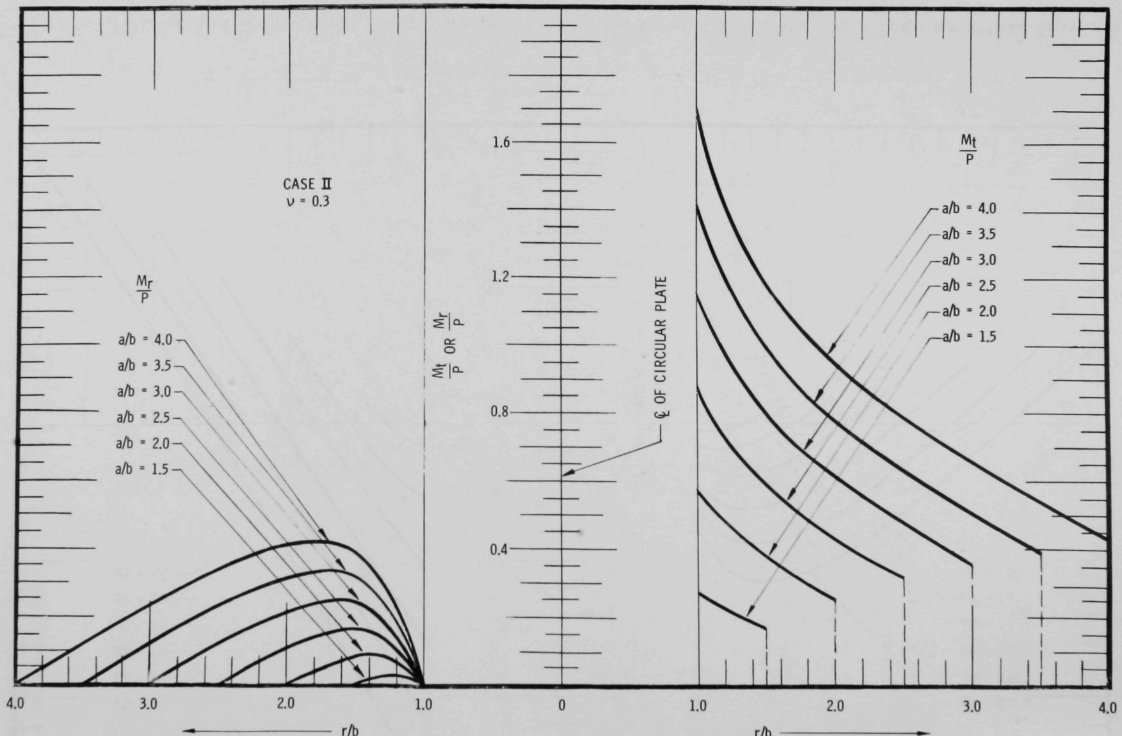


Fig. 6. Radial and Tangential Moments per Force Constant Diagram for Circular Plate Having Simply Supported Outer Edge and Free Inner Edge (Case II, $\nu = 0.3$)

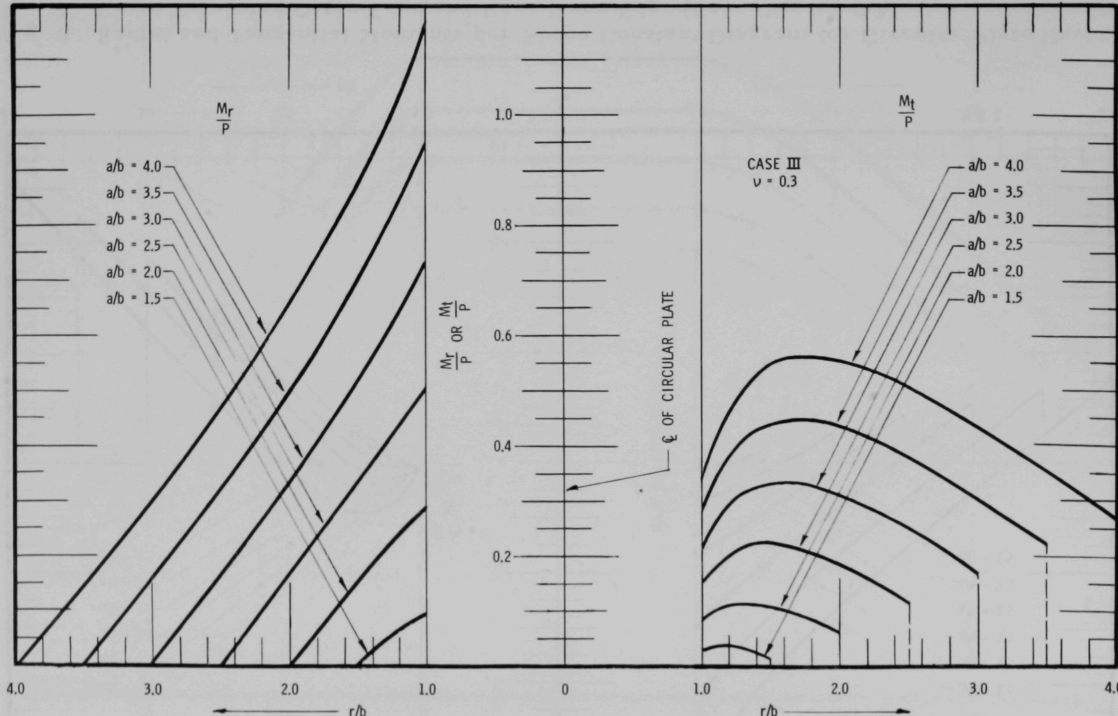


Fig. 7. Radial and Tangential Moments per Force Constant Diagram for Circular Plate Having Simply Supported Outer Edge and Fixed Inner Edge (Case III, $\nu = 0.3$)

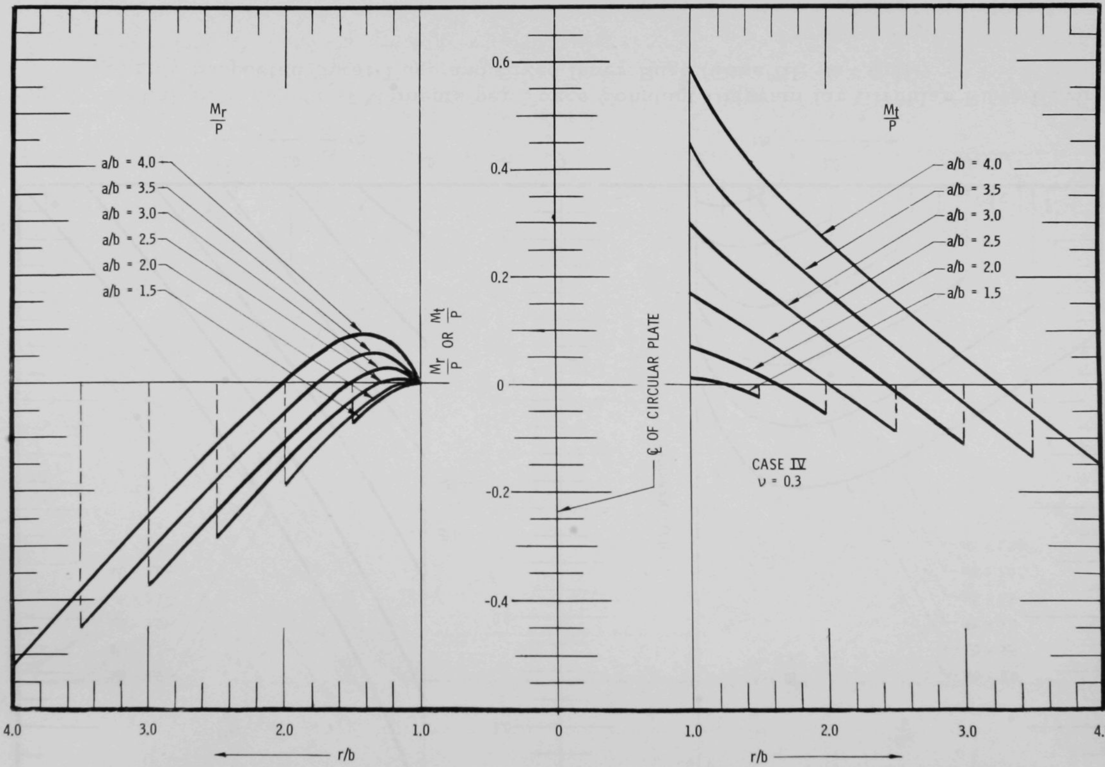


Fig. 8. Radial and Tangential Moments per Force Constant Diagram for Circular Plate Having Fixed Supported Outer Edge and Free Inner Edge (Case IV, $\nu = 0.3$)

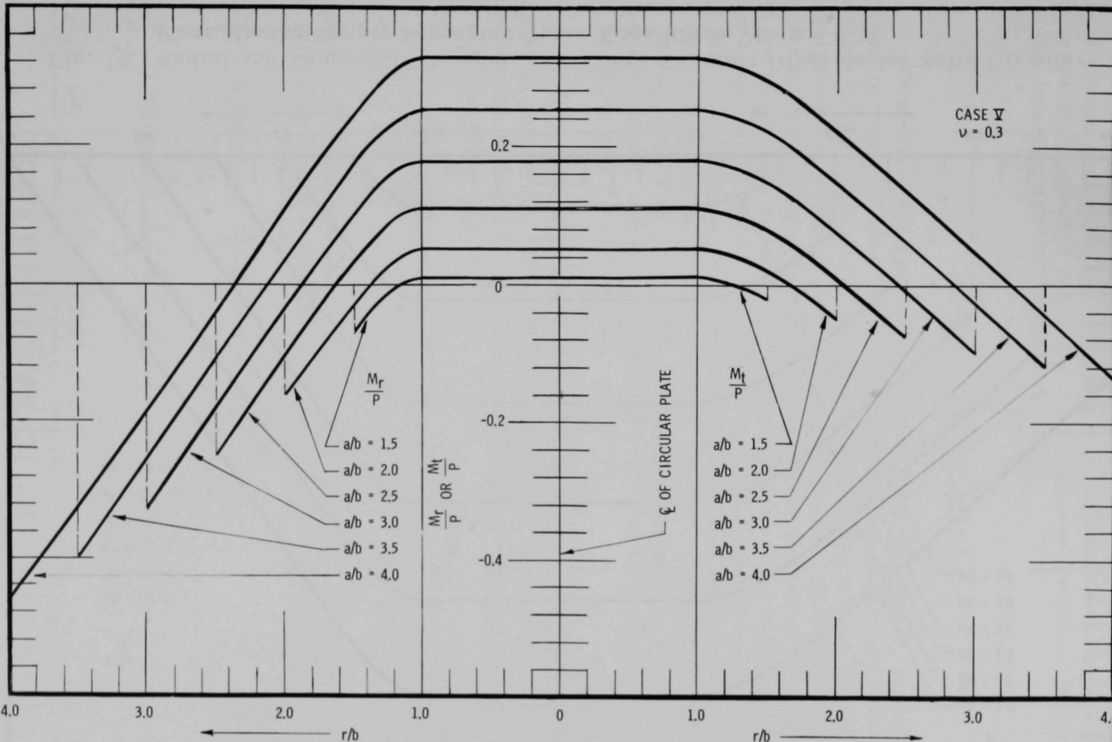


Fig. 9. Radial and Tangential Moments per Force Constant Diagram for Solid Circular Plate Having Fixed Supported Outer Edge (Case V, $\nu = 0.3$)

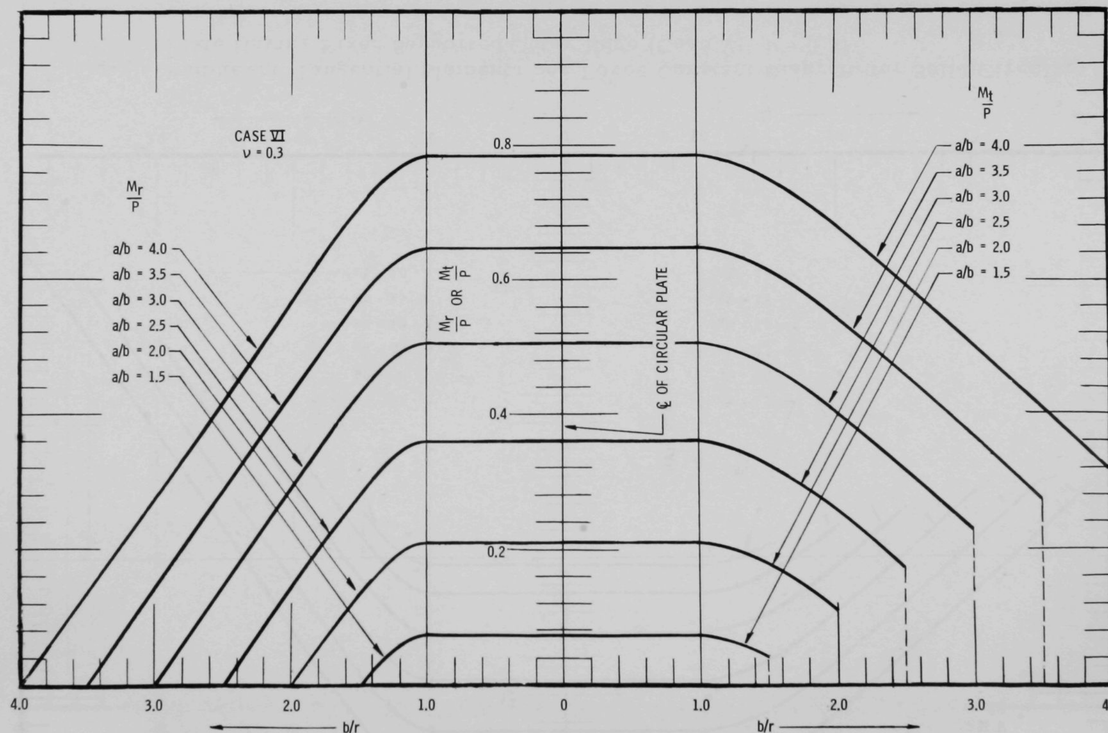


Fig. 10. Radial and Tangential Moments per Force Constant Diagram for Solid Circular Plate Having Simply Supported Outer Edge (Case VI, $\nu = 0.3$)

Calculations for plotting Figs. 5 through 10 were performed with the CDC 3600 computer using Argonne National Laboratory program 1837/PAD 143. For any given combination of values for ν and the ratio a/b, this program computes and tabulates the deflection constants and the radial and tangential moments per force in all six cases where r/b ranges from 1 to the selected a/b in increments of 0.1.

NUMERICAL EXAMPLE

Determine the optimum, uniform, plate thickness, the maximum bending moment, and the maximum bending stress of a symmetrical, variably loaded, flat, solid, circular, copper plate where the maximum permissible deflection is half the plate thickness. The variable load has the form P/r^2 acting on a surface bounded by circles of an inner radius and the outer edge support. Because of the construction of the outer end edge support, the plate is considered to be simply supported. Given plate and load specifications are: outer plate and load radius, $a = 40.85$ cm (16.083 in.); inner load radius, $b = 21.50$ cm (8.465 in.); and load constant, $P = 150\ kg_f$ (330.7 lb_f). The following mechanical properties apply for the specified copper: modulus of elasticity, $E = 10.55 \times 10^5\ kg_f/cm^2$ ($15.0 \times 10^6\ lb_f/in.^2$); and Poisson's ratio, $\nu = 0.33$.

Referring to the tabulated equations of Case VI, transposing the maximum deflection equation, and substituting the flexure rigidity expression into the transposed equation, the following equation is ascertained for the uniform plate thickness:

$$h = \left\{\frac{3\ Pa^2\ (1 - \nu^2)}{E} \left[\left(\frac{3 + \nu}{1 + \nu} + \frac{b^2}{a^2}\right)\ln\frac{a}{b} - \left(\frac{2 + \nu}{1 + \nu}\right)\left(1 - \frac{b^2}{a^2}\right)\right]\right\}^{1/4} . \quad (17)$$

From the maximum bending-moment equation and the rearranged equation, the following terms containing Poisson's ratio are first computed:

$$\frac{1 - \nu}{1 + \nu} = 0.50376; \qquad \frac{3 + \nu}{1 + \nu} = 2.50376;$$

$$\frac{2 + \nu}{1 + \nu} = 1.75188; \qquad \frac{1 + \nu}{4} = 0.33250;$$

$$\frac{3\,Pa^2(1 - \nu^2)}{E} = \frac{3\,(150\ kg_f)\,(40.85\ cm)^2(1 - \overline{0.33}^{\,2})}{10.55 \times 10^5\ kg_f/cm^2} = 0.63426\ cm^4. \quad (18)$$

To use Table I, the ratio of the inner load radius to the outer radius is

$$a/b = 40.85\ cm/21.50\ cm = 1.900. \quad (19)$$

Hence, using Eq. (17), computed terms, and Table I, the required thickness is

$$h = \{0.63426\ \text{cm}^4\ [(2.50376 + 0.27701)0.64185 - 1.75188(0.72299)]\}^{1/4}$$

$$= \{0.32870\}^{1/4} = 0.757\ \text{cm}\ (0.298\ \text{in.}). \qquad (20)$$

The maximum moment becomes

$$M_{max} = (150\ \text{kg}_f\ \text{cm/cm})(0.33250)[0.41197 + 0.50376(0.64185) - (1/2)(0.50376)(0.72299)] = 27.59\ \text{kg}_f\text{-cm/cm}. \qquad (21)$$

Using this obtained maximum moment, the maximum unit stress is

$$\sigma_{max} = \pm \frac{6\ M_{max}}{h^2} = \pm \frac{6\,(27.59\ \text{kg}_f\text{-cm/cm})}{(0.757\ \text{cm})^2}$$

$$= \pm 288.9\ \text{kg}_f/\text{cm}^2\,(4{,}110\ \text{lb}_f/\text{in.}^2). \qquad (22)$$

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