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INSIDE QUANTUM FIELD THEORY AND LECTURES ON NEUTRINO INTERACTIONS

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INSIDE QUANTUM FIELD THEORY AND LECTURES ON NEUTRINO INTERACTIONS

by

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December 1964

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PREFACE

The intense effort being devoted to high-energy physics these days has resulted more and more in the emergence of two distinct cultures, experimentalists and theoreticians. Experiments at many megavolts often cost some megabucks and sometimes seem to have a few mega-authors on the paper.

These lectures grew out of a visit to the Neutrino Group of the High Energy Physics Division of Argonne National Laboratory. This visit could possibly be construed as our effort to bridge the two-culture gap. As exchange for these lectures on theory, I was initiated into the mysterious rites - lasting many days and nights - that experimentalists practice around the ZGS.

I am indebted to Professor R. G. Sachs for extending the hospitality of the High Energy Physics Division, and to Drs. T. B. Novey and D. D. Jovanovic of the Neutrino Group for making the visit possible.

PART ONE
INSIDE QUANTUM FIELD THEORY

I. Introduction

The purpose of these lectures is to provide a review and summary of the fundamental ideas of quantum field theory, with a special emphasis on selecting those features that are relevant to applications, and merely acknowledging the features that are not. We are definitely interested in building a minimal set of ideas.

The first complication of the usual formalism of quantum field theory, which can be omitted, is second quantization. Ordinary quantum mechanics already provides a rule of associating a wave function to each particle, so we do not actually need an algebra to do it for us. Second quantization does give some interesting results, but the point is that it is not necessary. Creation and annihilation operators never appear in cross sections, and the original rule of quantum mechanics is sufficient. Quantum dynamics is fully contained in the wave equations.

II. Noninteracting Fields

From classical physics, we know that the momentum-energy relation for free particles is

$$p^2 - m^2 = 0. \tag{1}$$

The choice of units here is $\hbar = c = 1$; this makes mass proportional to length^{-1}. A real metric is chosen $p^2 \equiv p_\alpha p^\alpha \equiv p_0^2 - \underset{\sim}{p}^2$. The metric tensor is $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \equiv g^{\alpha\beta} \equiv g_{\alpha\beta}$. While the real metric has the complication compared to the complex metric of maintaining the covariant-contravariant distinction, it has the simplification of introducing no problems with complex conjugation.

If we use the usual momentum operator correspondence of quantum mechanics, $p_\alpha \to i \dfrac{\partial}{\partial x_\alpha} \equiv i\partial\alpha$, then Eq. (1) results in the following wave equation for spin zero (Klein-Gordon equation):

$$(\partial^2 + m^2)\,\psi(x) = 0. \tag{2}$$

How does one relate a probability density to this equation? Total probability for free particles, must be conserved; i.e., it is constant; in fact, it equals 1 with some normalization. Note that $(\partial^2 + m^2)\,\psi^*(x) = 0$; thus, $\psi^*(\partial^2+m^2)\,\psi - \psi(\partial^2+m^2)\,\psi^* = 0$. This is just $\partial^\alpha(\psi^*\partial_\alpha\psi - \psi\,\partial_\alpha\psi^*) = 0$. Then the (conserved) probability current is

$$J_\alpha \equiv i(\psi^* \partial_\alpha \psi - \psi \partial_\alpha \psi^*). \tag{3}$$

The factor i is put in to make J_α Hermitian; it also corresponds to $p_\alpha \to i\partial\alpha$.

The Dirac equation resulted from a search for a wave equation that was only first order in $i\partial\alpha$. The argument is as follows:

Every particle must satisfy $(\partial^2 + m^2)\,\psi(x) = 0$. Then, to find a first-order equation in $i\partial\alpha$, we can only have combinations such as $(i\gamma^\alpha\partial\alpha - m)\,\psi(x) = 0$, where the γ^α are constant (matrices), independent of x. Note that only m can occur, not m^2, m^3, etc., since $(\partial^2 + m^2)\,\psi = 0$; $\partial^2 + m^2$ can be obtained through multiplication by $(i\gamma^\beta\partial_\beta + m)$. We obtain

$$-(\gamma^\beta\gamma^\alpha\partial_\beta\partial_\alpha + m^2)\,\psi = 0. \tag{4}$$

We must require $\gamma^\beta\gamma^\alpha\partial_\beta\partial_\alpha \equiv \partial^\alpha\partial_\alpha$ to obtain the Klein-Gordon equation. Now $\partial_\beta\partial_\alpha = \partial_\alpha\partial_\beta$ is symmetric; thus, $\gamma^\beta\gamma^\alpha = (1/2)(\gamma^\beta\gamma^\alpha + \gamma^\alpha\gamma^\beta) + (1/2)(\gamma^\beta\gamma^\alpha - \gamma^\alpha\gamma^\beta)$, contracted with $\partial_\beta\partial_\alpha$, leaves only the symmetric part; further, $\partial_\alpha\partial^\alpha = \partial_\alpha\partial_\beta g^{\alpha\beta}$. Therefore,

$$\gamma^\beta\gamma^\alpha + \gamma^\alpha\gamma^\beta = 2g^{\alpha\beta}. \tag{5}$$

Since the γ^α anticommute, they cannot be just numbers but must be matrices. The smallest set of four matrices that satisfy the anticommutation relations (5) are 4 x 4; they correspond to spin 1/2. This means $\psi(x)$ is a 1 x 4 spinor. The conclusion is that if the γ^α satisfy Eq. (5), we have the wave equation for spin one-half,[1]

$$(i\gamma^\alpha\partial_\alpha - m)\,\psi(x) = 0. \tag{6}$$

The Hermitian conjugate (complex conjugate, transpose) satisfies $-i\partial_\alpha\psi^\dagger\gamma^{\alpha\dagger} - \psi^\dagger m = 0$. Equation (5) is the only restriction upon the γ^α made by quantum dynamics. The relation between γ^α and $\gamma^{\alpha\dagger}$ can be determined by convenience. We choose $\gamma_0^\dagger = \gamma_0$ and $\underset{\sim}{\gamma}^\dagger = -\underset{\sim}{\gamma}(\gamma_5^\dagger = -\gamma_5)$. This choice determines $\overline{\psi} = \psi^\dagger\gamma_0$ as the adjoint spinor so that $-i\partial_\alpha\overline{\psi}\gamma^\alpha - \overline{\psi}m = 0$.[2]

---

[1]Other combinations of the γ-matrices are useful in formulating interactions; $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_0$ anticommutes with the γ_α; $\gamma_5\gamma_\alpha + \gamma_\alpha\gamma_5 = 0$. For magnetic moments, the antisymmetric $\sigma_{\alpha\beta} = (\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)/2$ is the appropriate combination.

[2]In general, the only condition one would make upon the adjoint spinor is $-i\partial_\alpha\overline{\psi}\gamma^\alpha - \overline{\psi}m = 0$; i.e., $\overline{\psi}(\gamma \cdot p - m) = 0$, if $(\gamma \cdot p - m)\,\psi = 0$. If we suppose $\overline{\psi} = \psi^\dagger A$, then $-i\partial_\alpha\psi^\dagger A\, A^{-1}\gamma^{\alpha\dagger}A - \psi^\dagger Am = 0$ or $A^{-1}\gamma^{\alpha\dagger}A = \gamma^\alpha$. For $\gamma_0^\dagger = \gamma_0$, $\gamma_i^\dagger = -\gamma_i$, $i = 1, 2, 3$, then $A = \gamma_0$. Note also that $\gamma_\alpha^\dagger = \gamma_\alpha$ would yield $A = 1$. As $A = \gamma_0$ is conventional, for historic reasons we will use $\overline{\psi} = \psi^\dagger\gamma_0$.

To construct the probability, we have $\bar{\psi}(i\gamma^{\alpha}\partial_{\alpha} - m)\,\psi - [-i\partial_{\alpha}\bar{\psi}\gamma^{\alpha}\psi - m\bar{\psi}\psi] = 0$, which yields $\partial_{\alpha}(\bar{\psi}\gamma^{\alpha}\psi) = 0$; thus the (conserved) probability current for spin 1/2 is

$$J_{\alpha} = \bar{\psi}(x)\;\gamma_{\alpha}\psi(x). \tag{7}$$

Usually one is interested in solutions of the wave equations that correspond to definite momentum. For the Klein-Gordon equation, the plane wave solution is

$$\psi_p(x) = e^{-ip\cdot x}. \tag{8}$$

We can check the normalization of this solution by computing $J_{\alpha}(x)$,

$$J_{\alpha}(x) = 2p_{\alpha}. \tag{9}$$

$J_0 = 2E$ as the covariant normalization corresponds to 2E particles per unit volume, rather than the one particle per unit volume of nonrelativistic quantum mechanics. The spin 1/2 plane wave solutions are

$$\psi_p(x) = u_p e^{-ip\cdot x}, \tag{10}$$

where u_p is a 1 x 4 spinor satisfying

$$(\gamma\cdot p - m)\;u_p = 0. \tag{11}$$

To determine the most useful normalization for the $\bar{u}_p$, we again examine $J_{\alpha}(x) = \bar{u}_p\gamma_{\alpha}u_p$. We would like to have something like $J_{\alpha} \sim 2p_{\alpha}$. Note that $(\gamma\cdot p/m)\;u_p = u_p$, and $\bar{u}_p(\gamma\cdot p/m) = \bar{u}_p$. Thus,

$$J_{\alpha} = \frac{\bar{u}_p(\gamma_{\alpha}\gamma\cdot p + \gamma\cdot p\gamma_{\alpha})\;u_p}{2m} = 2p_{\alpha}\frac{\bar{u}_p u_p}{2m}.$$

Thus the appropriate normalization for spinors is

$$\bar{u}_p u_p = 2m. \tag{12}$$

The wave functions for antiparticles are simply related to those for particles. For spin zero, an antimeson would have a wave function $e^{+ip\cdot x}$ instead of $e^{-ip\cdot x}$. For spin 1/2, we would use $\bar{u}(-p)\;e^{+ip\cdot x}$ instead of $u(p)\;e^{-ip\cdot x}$ for the antifermion of momentum p. In this case, we would have the normalization $\bar{u}(-p)\;u(-p) = -2m$.[3]

[3] Our choice of wave functions for antiparticles corresponds, in a second quantized formalism, to the fact that a field operator $\psi(x)$ is a sum of two parts, one of which annihilates particles and the other creates antiparticles; $\bar{\psi}$ then creates particles and annihilates antiparticles.

Since we are interested in interacting particles we need to discuss inhomogeneous equations, that is, the equations appropriate when sources are present. The Green's function or propagator or unit-point-source solution for the Klein-Gordon equation satisfies[4]

$$-(\partial^2 + m^2)\, G(x_\alpha, x'_\alpha) = \delta^{(4)}(x_\alpha - x'_\alpha).$$

Taking the Fourier transforms

$$G(x_\alpha - x'_\alpha) = \int \frac{d^4p}{(2\pi)^4}\, G(p) e^{-ip(x - x')},$$

and

$$\delta^{(4)}(x_\alpha - x'_\alpha) = \int \frac{d^4p}{(2\pi)^4}\, 1\, e^{-ip(x - x')},$$

we have $(p^2 - m^2)\, G(p) = 1$, or

$$G(p) = \frac{1}{p^2 - m^2}, \tag{13}$$

which is the propagator for spin zero.[5] For spin 1/2, we have $(\gamma \cdot p - m)$ $S(p) = 1$, where $S(p)$ is a 4 x 4 Green's matrix, and 1 is the 4 x 4. The propagator for spin 1/2 is

$$S(p) = \frac{1}{\gamma \cdot p - m} = \frac{\gamma \cdot p + m}{p^2 - m^2}. \tag{14}$$

---

[4]This is true because we can translate the origin to x'_α, $G(x, x') = G(x - x')$. Actually $G(x, x') = G(x', x)$, hence can depend only on the magnitude of $x_\alpha - x'_\alpha$. To see this, we take $-G(x', x)\left(\frac{\partial^2}{\partial x^2} + m^2\right)$ $G(x, x') = \delta(x - x')\, G(x', x)$, and integrate twice by parts. Since the surface terms at ∞ vanish due to the boundary conditions on G, we obtain $-\left(\frac{\partial^2}{\partial x^2} G(x', x) + m^2 G(x', x)\right) G(x, x') = \delta(x - x')\, G(x', x)$. If we divide by $G(x, x')$, the ratio $G(x', x)/G(x, x')$ contributes to the right-hand side only for $x = x'$ when it is 1. Thus, $-\left(\frac{\partial^2}{\partial x^2} + m^2\right)$ $G(x', x) = \delta(x - x')$. Hence, $G(x, x') = G(x', x)$. This is an example of symmetry in G which follows from the self-adjoint property of ∂^2.

[5]It is more conventional to denote the Klein-Gordon Green's function as $\Delta(x - x')$, and for zero mass $D(x - x')$. A prescription is needed for circumventing the poles in $(p^2 - m^2)^{-1}$ when the inversion to configuration space is performed. The casual (or Feynman) propagators D_F, Δ_F, and S_F result from the prescription $(p^2 - m^2 + i\epsilon)^{-1}$.

Quite generally, propagators have the form

$$\frac{\text{spin sum}}{p^2 - m^2}.$$

This results from the fact that the wave functions form a complete set for an appropriate subspace of spin space, so the inhomogeneous equation, which has a 1 for zero spin, has generally the form $(p^2 - m^2)\, G = \sum\limits_{\text{spins}} u\bar{u}$.

For the Dirac equation, we have the useful by-product from determination of the propagator,

$$\sum_{\text{spin}=1}^{2} u\bar{u} = \gamma \cdot p + m. \tag{15}$$

For antifermions, the spin sum would be

$$\sum_{r=1}^{2} u_r(-p)\, \bar{u}_r(-p) = p \cdot \gamma - m.$$

These spin sums correspond to the positive and negative energy projection operators times $\bar{u}u$ as follows:

$$\frac{\gamma \cdot p + m}{2m}\,(2m) \quad \text{and} \quad \frac{-\gamma \cdot p + m}{2m}\,(-2m), \text{ respectively.}$$

III. <u>Interacting Fields</u>

The inclusion of interaction in this formalism is most easily done by using the Lagrangian. From the equations of motion (2) and (6), the free or <u>noninteracting Lagrangian density for spin zero</u> is

$$\mathcal{L}_0 = +\partial^\alpha \psi^*(x)\, \partial_\alpha \psi(x) - m^2 \psi^*(x)\, \psi(x), \tag{16}$$

where ψ and ψ^* are regarded as independent in the variation.

For the Dirac equation,

$$\mathcal{L}_0 = \overline{\psi}(x)(-i\gamma^\alpha \partial_\alpha + m)\, \psi(x), \tag{17}$$

and ψ and $\overline{\psi}$ are regarded as independent spinor fields in the variations. An interaction term in the Lagrangian will usually be of the form of some non-quadratic product (interaction implies that the equations of motion be non-linear; otherwise two solutions could be superimposed) of fields with a coupling constant in front, which is a measure of the strength of the interactions. Some examples are:

1. Self-interaction of a neutral scalar field, $\mathcal{L}_I = g\phi^3(x)$.

2. Interaction of an electron with the electromagnetic field, $\mathcal{L}_I = e\bar{\psi}(x)\,\gamma_\alpha\psi(x)\;A^\alpha(x)$.

   This last comes from the principle of minimal electromagnetic interaction, $p_\alpha \rightarrow p_\alpha - eA_\alpha$; thus, $\bar{\psi}(-i\gamma^\alpha\partial_\alpha)\,\psi \rightarrow \bar{\psi}(-i\gamma^\alpha\partial_\alpha)\,\psi + e\bar{\psi}\gamma^\alpha\psi A_\alpha$.

3. The weak interactions of nucleons and leptons, $\mathcal{L}_I = \frac{G}{\sqrt{2}}\,\bar{\psi}_p(x)\,\gamma_\alpha(1+\lambda i\gamma_5)\,\psi_n(x)\;\bar{\psi}_\ell(x)\,\gamma^\alpha(1+i\gamma_5)\,\psi_\nu(x)$.

Our next task is to develop rules for constructing the transition matrix elements or amplitudes. We will consider these amplitudes from several points of view. First, what would we expect as the basis of nonrelativistic perturbation theory? Recall that nonrelativistically, cross sections are related by

$$\sigma = \frac{1}{\text{flux}}\,|T|^2 d\rho,$$

where the density of final states of $d\rho$ is $d^3p/(2\pi)^3$ for each particle, with an overall constraint of energy and momentum conservation. A typical second-order T resembles

$$T_{fi} = \sum_n{}' \frac{H^*_{fn}H_{ni}}{E_n - E_i}.$$

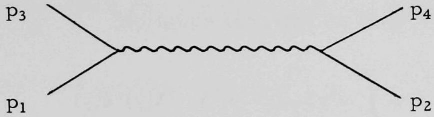
We might expect the generalization of $H_{ni} = \int d^3x\,H$ to be $\int d^4x\,\mathcal{L}$. As for $(E_n - E_i)^{-1}$, this resembles $(p^2 - m^2)^{-1}$. The resemblance is clearer if we note

$$\frac{1}{p^2 - m^2} = \frac{1}{2E}\left[\frac{1}{E - (P^2+m^2)^{1/2}} + \frac{1}{E + (P^2+m^2)^{1/2}}\right].$$

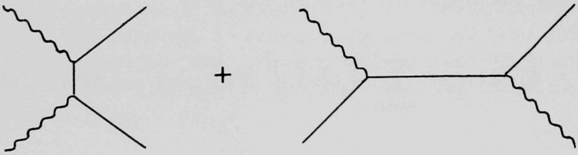
So we just have a sum of energy denominators because both positive and negative energies contribute, i.e., particle and antiparticle. As an example, we would expect the amplitude for, say, electromagnetic electron-electron scattering to resemble

$$T = (-4\pi i)\;e^2\,\frac{\bar{u}(p_3)\,\gamma^\alpha u(p_1)\;\bar{u}(p_4)\,\gamma_\alpha u(p_2)}{(p_1 - p_3)^2},$$

where the constants $(-4\pi i)$, which we will add to our rules later, have been put in for completeness. Note that $g^{\alpha\beta}/(p_1 - p_3)^2$ is the photon propagator. We can draw the following picture of this process (Feynman diagram):

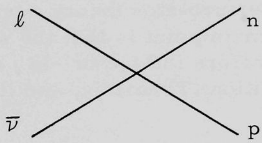


Each element of the amplitude corresponds to an element of the Feynman diagram. Each of the external electron lines corresponds to a wave function u or $\bar{u}$. Each internal line corresponds to a propagator or "energy denominator," and each vertex to the basic interaction Lagrangian. The diagrams are arranged in an approximate space-time order. Since these diagrams have such a one-to-one correspondence to matrix elements, they are extremely useful as a means of describing particular matrix elements. For example, a theorist will often say, "I calculated ...."



which corresponds to lowest-order Compton scattering,

or



which portrays neutrino interactions.

The convention I use for diagrams is ———————— for a Fermion, ------------- for spin zero, ∼∼∼∼∼∼∼∼ for spin one, and a double line ════════ for most other possibilities.

Suppose for a moment, we discuss the transition amplitude in a more formal way (for this example, quantum electrodynamics of Fermions).

The equations of motion are

$$(i\gamma \cdot p - m)\, \psi(x) \;=\; e\gamma \cdot A(x)\, \psi(x). \tag{18}$$

Even though all calculation will be done in momentum space, it is easiest to develop the rules in configuration space since scattering involves a definite time direction.

Make Eq. (18) into an integral equation,

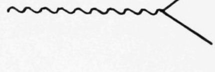
$$\psi(x) = \psi_0(x) + e \int d^4y S_F(x-y)\, \gamma \cdot A(y)\, \psi(y). \tag{19}$$

A solution of the homogeneous equation for $\psi_0(x)$ must be added in the inversion of a differential into an integral operator. The Green's function $S_F(x-y)$ satisfies $(i\gamma\cdot\partial - m)\, S_F(x-y) = \delta^{(4)}(x-y)$ and is just the Fourier transform of S(p). In the limit $x_0 \to +\infty$, we have $(i\gamma\cdot\partial - m)\, S_F = 0$. Recalling the relation between Green's function and spin sum, we have $S_F(x-y) \to i \sum_{\text{spin}} \psi_{0f}(x)\, \bar{\psi}_{0f}(y)$, where ψ_{0f} is a final free Dirac ψ. Thus for $x_0 \to +\infty$, we have the following form of Eq. (19):

$$\psi_f(x) = \psi_{0i}(x) - ie \sum_{\text{spin}} \psi_{0f}(x) \int d^4y \bar{\psi}_{0f}(y)\, \gamma \cdot A(y)\, \psi_i(y), \tag{20}$$

where ψ_{0i} is the incident plane wave, and ψ_i and A are the exact incident ψ_i (plane wave and scattered wave) and the exact electromagnetic potential, respectively. Perturbation theory results from an iterative solution of Eq. (20). The main point is that the second term is the change in the wave function. Therefore the factor $-ie \int d^4y \bar{\psi}_{0f}\, \gamma \cdot A\psi_i$, with exact states ψ_i, is just the transition (T) matrix, and it has the same form as the Lagrangian.

This Lagrangian corresponds to the following diagram:



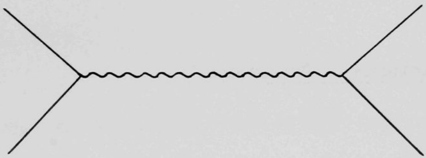
If we have initially two particles present and if we iterate Eqs. (19) and (20) plus the corresponding equation for the electromagnetic field,

$$A_\alpha(x) = A^0_\alpha(x) + 4\pi e \int d^4y G_{\alpha\beta}(x-y)\, \bar{\psi}_f(g)\, \gamma^\beta \psi_i(y), \tag{21}$$

where the Fourier transform of $G_{\alpha\beta}(x - y)$ is $G_{\alpha\beta}(k) = g_{\alpha\beta}/k^2$, we obtain the matrix element,

$$-i4\pi e^2 \int d^4y \int d^4x \bar{\psi}_{1f}(x)\, \gamma^{\alpha}\psi_{1i}(x)\, G_{\alpha\beta}(x - y)\, \bar{\psi}_{2f}(y)\, \gamma^{\beta}_{2i}(y).$$

This corresponds to the following diagram:



IV. Rules for Calculation

The basic technique of calculation of T matrix elements is reduced to one of drawing the relevant Feynman diagrams and then using the rules to associate wave functions, etc., to each part of each diagram. The rules are listed below (the 4π that enters into the rules for electrodynamic interaction comes from $\partial^2 A_{\alpha} = 4\pi J_{\alpha}$).

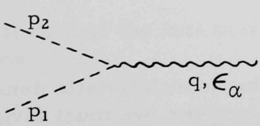
Propagators

$\dfrac{1}{p^2 - m^2}$ Spin zero

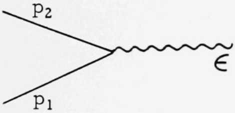
$\dfrac{\gamma \cdot p + m}{p^2 - m^2}$ Spin 1/2

$\dfrac{-ig_{\mu v}}{k^2}$ Photon

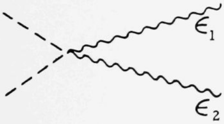
Electromagnetic Interaction



$-i\sqrt{4\pi}\, e(p_1 + p_2) \cdot \epsilon$ Spin zero

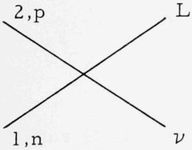


$\sqrt{4\pi}\ e\bar{u}(p_2)\epsilon \cdot \gamma u(p_1)$ Spin 1/2

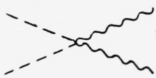


$2i4\pi\ e^2 \epsilon_1 \cdot \epsilon_2$ The $A(x)^2$ interaction for spin zero.

Weak Interactions of Nucleons



$$\frac{G}{\sqrt{2}}\,\bar{u}_p(p_2)\ \gamma_\alpha(1+\lambda i\gamma_5)\ u_n(p_1)\ \bar{u}_L(p_L)\ \gamma^\alpha(1+i\gamma_5)\ u_\nu(p_\nu)$$

For the electromagnetic amplitudes, the factors of $-i4\pi$ that we found in the second-order amplitude have been associated with the various parts in ways that are especially convenient for calculation, i.e., rationalized units. The factor 2 in  results from the two ways in which the photon wave functions can be assigned.

The particular factors of i and $\sqrt{4\pi}$ in these rules are choices that are especially convenient and were first worked out this way by Feynman. They are given in his lectures, Theory of Fundamental Processes. These rules actually give $i\langle|T|\rangle$ as the matrix element.

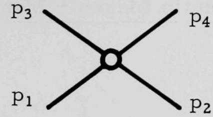
In a cross section, there are two factors in addition to $|\langle|T_{if}|\rangle|^2$. First there is the flux factor. Recall that we have a normalization of 2E per unit volume; thus the flux of incoming particles is given by

$$f = 2E_{i1}2E_{i2}v_{12}, \tag{22}$$

where v_{12} is the relative velocity.

Phase space is the last ingredient. The nonrelativistic density of states would be $d^3p/(2\pi)^3$ for each particle. However, we must divide out our relativistic normalization; hence for each particle we have $d^3p/[2E(2\pi)^3]$. If we note that $\int dE\,\delta(E^2-p^2-m^2) = (2E)^{-1}\int dE\,\delta\left(E-\sqrt{p^2+m^2}\right)$, then we have the manifestly covariant form $2\pi\ \delta(p^2-m^2)\ d^4p/(2\pi)^4$ for the density of states of each particle. The overall density of states includes a δ function for momentum and energy conservation.

For the cross section for a two-body process,



we then summarize the rules

$$d\sigma = \frac{1}{f} \left|\langle p_3 p_4 | T | p_1 p_2 \rangle\right|^2 d\rho_2; \qquad f = 2E_1 2E_2 v_{12};$$

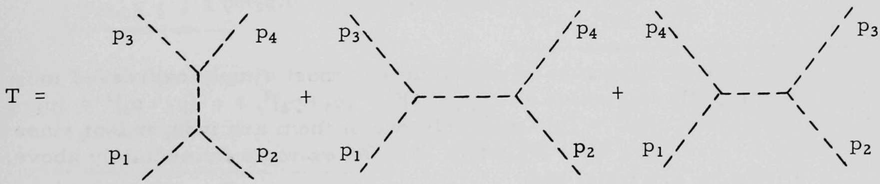
$$d\rho_2 = (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - p_3 - p_4)\; 2\pi\delta(p_3^2 - m_3^2)\; 2\pi\delta(p_4^2 - m_4^2)\; \frac{d^4p_3\, d^4p_4}{(2\pi)^4 (2\pi)^4}. \tag{23}$$

If constraints are counted then for ρ_2 there are eight variables and six constraints. One of the two remaining variables of integration is an azimuthal angle, and usually one is dealing with a situation of azimuthal symmetry. This means that there is ordinarily only one variable in a two-body process. This may be taken as an angle, or as an energy, as is appropriate. For ρ_3 there are five variables remaining after satisfaction of the constraints. Four remain after elimination of the azimuthal angle. The flux and phase space appropriate for neutrino interactions is covered in Section I of Part Two.

V. Examples of $\langle|T|\rangle$ and $\overline{|\langle|T|\rangle|^2}$

A. Bosons

1. Lowest-order Scattering in ϕ^3 Theory



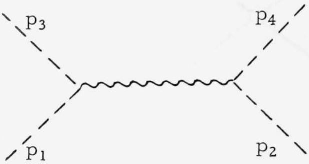
Note that both of the last pair must be present because of Bose-Einstein statistics.

$$\langle p_3 p_4 | T | p_1 p_2 \rangle = \frac{1}{(p_1 + p_2)^2 - m^2} + \frac{1}{(p_1 - p_3)^2 - m^2} + \frac{1}{(p_1 - p_4)^2 - m^2}.$$

Using $p_1 + p_2 = p_3 + p_4$, the various combinations can be changed somewhat.[6]

2. <u>Electromagnetic Meson ± Meson Scattering</u>

T =



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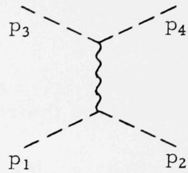
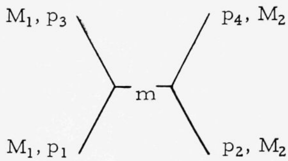


$$T = -i4\pi e^2\left[\frac{(p_1+p_3)\cdot(p_2+p_4)}{(p_1-p_3)^2} + \frac{(p_1-p_2)\cdot(p_3-p_4)}{(p_1+p_2)^2}\right].$$

B. <u>Fermions</u>

First let us treat a simple example, proton-neutron scattering through a scalar field. The vertex is $g\,\bar{u}u$. Thus,

T =



$$= g^2\,\frac{\bar{u}_p(p_3)\,u(p_1)\,\bar{u}(p_4)\,u(p_2)}{(p_1-p_3)^2 - m^2};$$

$$|T|^2 = \frac{g^4}{[(p_1-p_3)^2 - m^2]^2}\,|\bar{u}(p_3)\,u(p_1)|^2|\bar{u}(p_4)\,u(p_2)|^2.$$

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[6] The analytic structure of amplitudes is most simply expressed in terms of the variables $s = (p_1+p_2)^2 = (p_3+p_4)^2$, $t = (p_1-p_3)^2 = (p_2-p_4)^2$, and $u = (p_1-p_4)^2 = (p_2-p_3)^2$; only two of them are independent since $s + t + u = M_1^2 + M_2^2 + M_3^2 + M_4^2$. For the example immediately above, all the M_i are equal and

$$\langle p_3p_4|T|p_1p_2\rangle = \frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2};$$

the amplitude has poles in s, t, and u. The physical significance of these variables is: they equal (total energy in center of mass)^2 for some particular process.

Since polarizations do not interest us, we want to average over the initial spins and sum over the final spins.

Thus,

$$\overline{|T|^2} = \frac{g^4}{[(p_1 - p_3)^2 - m^2]^2} \frac{1}{2} \sum_{\text{spins}} |\bar{u}(p_3)\, u(p_1)|^2 \frac{1}{2} \sum_{\text{spins}} |\bar{u}(p_4)\, u(p_2)|^2;$$

$$\begin{aligned}
\sum_{\text{spins}} |\bar{u}(p_3)\, u(p_1)|^2 &= \sum_{\text{spins}} \bar{u}(p_3)\, u(p_1)\, \bar{u}(p_1)\, u(p_3) \\
&= \sum_{\text{spins}} \bar{u}(p_3)(\gamma \cdot p_1 + M_1)\, u(p_3) \\
&= \sum_{\text{spin}} \sum_{ij} \bar{u}_i(p_3)(\gamma \cdot p_1 + M_1)_{ij}\, u_j(p_3) \\
&= \sum_{\text{spin}} \sum_{ij} (\gamma \cdot p_1 + M_1)_{ij}\, u_j(p_3)\, \bar{u}_i(p_3) \\
&= \sum_{ij} (\gamma \cdot p_1 + M_1)_{ij}\, (\gamma \cdot p_3 + M_1)_{ji} \\
&= \text{Trace}\,[(\gamma \cdot p_1 + M_1)(\gamma \cdot p_3 + M_1)].
\end{aligned}$$

Since we are dealing with 4 x 4 matrices,

$$\boxed{\text{Trace } 1 = 4.}$$

Define

$$\gamma \cdot A \equiv \tilde{A}.$$

$$\boxed{\text{Trace}\,[\tilde{A}, \ldots \tilde{A}_{2n+1}] = 0,}$$

i.e., the trace of an odd number of γ matrices vanishes. To show this,

$$\begin{aligned}
\text{Trace}\,[\tilde{A}_1 \ldots \tilde{A}_{2n+1}] &= \text{Trace}\,[\tilde{A}_1 \ldots \tilde{A}_{2n+1} i\gamma_5 i\gamma_5], \text{ since } (i\gamma_5)^2 = 1 \\
&= (-1)^{2n+1}\, \text{Trace}\,[i\gamma_5 \tilde{A}_1 \ldots \tilde{A}_{2n+1} i\gamma_5] \\
&= (-1)^{2n+1}\, \text{Trace}\,[\tilde{A}_1 \ldots \tilde{A}_{2n+1} i\gamma_5 i\gamma_5],
\end{aligned}$$

since matrices can be cyclically permuted within a trace. The factor $(-1)^{2n+1} = -1$; hence, the trace must be zero.

Trace $[\tilde{A}\tilde{B}] = 2A \cdot B$ Trace 1 - Trace $[\tilde{B}\tilde{A}]$, using the commutation relations. But Trace $[\tilde{B}\tilde{A}]$ = Trace $[\tilde{A}\tilde{B}]$. Hence,

$$\boxed{\text{Trace } \tilde{A}\tilde{B} = 4A \cdot B.}$$

From now on let us drop the word trace, and let the square bracket indicate when a trace is to be taken.

$$[\tilde{A}\tilde{B}\tilde{C}\tilde{D}] = 2A \cdot B\,[\tilde{C}\tilde{D}] - [\tilde{B}\tilde{A}\tilde{C}\tilde{D}]$$

$$[\tilde{B}\tilde{A}\tilde{C}\tilde{D}] = 2A \cdot C\,[\tilde{B}\tilde{D}] - [\tilde{B}\tilde{C}\tilde{A}\tilde{D}]$$

$$[\tilde{B}\tilde{C}\tilde{A}\tilde{D}] = 2A \cdot D\,[\tilde{B}\tilde{C}] - [\tilde{B}\tilde{C}\tilde{D}\tilde{A}]$$

$$[\tilde{B}\tilde{C}\tilde{D}\tilde{A}] = [\tilde{A}\tilde{B}\tilde{C}\tilde{D}]$$

If we add these up and use $[\tilde{A}\tilde{B}] = 4A \cdot B$,

$$\boxed{[\tilde{A}\tilde{B}\tilde{C}\tilde{D}] = 4(A \cdot B\,C \cdot D - A \cdot C\,B \cdot D + A \cdot D\,B \cdot C).}$$

Since $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_0$, its trace will vanish with any number of γ matrices less than four:

$$[i\gamma_5\tilde{A}\tilde{B}\tilde{C}\tilde{D}] = -4i\epsilon_{\alpha\beta\gamma\delta}A^{\alpha}B^{\beta}C^{\gamma}D^{\delta},$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the four-dimensional alternating symbol.

In a similar fashion, if the commutation relations are used, higher traces can be evaluated. Back to our example,

$$\text{Trace } [(\tilde{P}_1 + M_1)(\tilde{P}_3 + M_1)] = 4(M_1^2 + P_1 \cdot P_3),$$

and

$$\overline{|\langle|T|\rangle|^2} = \frac{g^4(M_1^2 + P_1 \cdot P_3)(M_2^2 + P_2 \cdot P_4)}{[(p_1 - p_3)^2 - m^2]^2}.$$

PART TWO

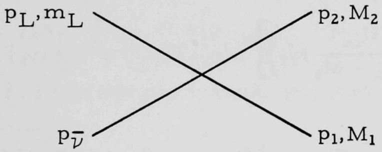
LECTURES ON NEUTRINO INTERACTIONS

I. Neutrino Interactions*

The weak interaction amplitude has the form

$$\langle |T| \rangle = \frac{G}{\sqrt{2}} J_\alpha^\dagger J^\alpha. \tag{24}$$

The Feynman diagram is



In momentum space, we know the Lepton current is

$$J_\lambda^L = \bar{u}(-p_\nu)\,\gamma_\lambda(1 + i\gamma_5)\,u(-p_L). \tag{25}$$

For the baryon current, consider the most general possibility,

$$J_\lambda^{B^\dagger} = \bar{u}(p_2)\Bigg[\gamma_\lambda(G_V + G_A i\gamma_5) + \sigma_{\lambda\beta}q^\beta\left(\frac{F_V}{M_1 + M_2} - \frac{F_A}{M_1 - M_2}\,i\gamma_5\right)$$

$$+ q_\lambda\left(\frac{-H_V}{M_1 + M_2} + \frac{H_A}{M_1 - M_2}\,i\gamma_5\right)\Bigg]\,u(p_1). \tag{26}$$

Why are there six form factors? There are two momenta p_1 and p_2, and also a spin, say, γ_α; the current is a four-vector, and only three independent four-vectors can be made from p_1, p_2, and γ_α. These, plus three more for axial vector, make six. The combinations $j'_\alpha \equiv (p_1 + p_2)_\alpha$, $q_\alpha \equiv (p_1 - p_2)_\alpha$, and γ_α are especially convenient. Using the Dirac equation, we can reduce $\bar{u}(p_2)\,\sigma_{\alpha\beta}q^\beta\,u(p_1)$ (and any other combinations) to these:

$$\bar{u}(p_2)\,\sigma_{\alpha\beta}q^\beta u(p_1) = \bar{u}(p_2)\left[\frac{\gamma_\alpha\tilde{p}_1 - \tilde{p}_1\gamma_\alpha - \gamma_\alpha\tilde{p}_2 + \tilde{p}_2\gamma_\alpha}{2}\right]u(p_1)$$

$$= \bar{u}(p_2)\Big[-(p_{1\alpha} + p_{2\alpha}) + (\gamma_\alpha\tilde{p}_1 + \tilde{p}_2\gamma_\alpha)\Big]\,u(p_1)$$

$$= \bar{u}(p_2)\Big[-j'_\alpha + (M_1 + M_2)\,\gamma_\alpha\Big]\,u(p_1). \tag{27}$$

This reduction of $\sigma_{\alpha\beta}$ is useful in performing traces.

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*Much of this section is based on a paper of mine: Nuovo Cimento **31**, 447 (1964).

The form factors, G, F, and H are generally functions, not constants. Time-reversal invariance implies that they are real. One of the basic assumptions of weak interactions is that of local currents; these are of the form $\bar{\psi}(x)\, O\psi(x)$. For plane-wave states, the x dependence is $e^{-(p_1 - p_2)\cdot x} \equiv e^{-iq\cdot x}$. If there is additional structure due to strong interactions, it smears out; i.e., it integrates over x, resulting in functions of q^2 only. Thus the form factors are functions of q^2, in general.

The transition amplitude for neutrino interactions is

$$\langle p_2 p_L |T| p_1 p_\nu \rangle = \frac{1}{\sqrt{2}}\bar{u}(p_2)\left[\gamma_\lambda(G_V + G_A i\gamma_5) + \sigma_{\lambda\beta}q^\beta\left(\frac{F_V}{M_1 + M_2} + \frac{F_A}{M_1 - M_2}i\gamma_5\right)\right.$$
$$\left. + q_\lambda\left(\frac{H_V}{M_1 + M_2} + \frac{H_A}{M_1 - M_2}i\gamma_5\right)\right] u(p_1)\, \bar{u}(p_L)\, \gamma^\lambda(1 + i\gamma_5)\, u(p_\nu). \tag{28}$$

The H terms can be ignored since $q_\lambda \bar{u}(p_L)\gamma^\lambda(1 + i\gamma_5)\, u(p_\nu) = \bar{u}(P_L)(\tilde{P}_L - \tilde{P}_\nu)(1 + i\gamma_5)\, u(p_\nu) = m_L \bar{u}(p_L)(1 + i\gamma_5)\, u(p_\nu)$.

When the amplitude is squared and summed over spins, all terms involving H have factors of m_L^2 in them. Even for muons, m_L is so small compared to the other masses present that these H terms contribute at most a few percent to the cross section.

The axial magnetism term, $F_A \sigma_{\lambda\beta} q^\beta i\gamma_5$, can presumably be ignored also, but for a different reason. If $\bar{u}\sigma_{\lambda\beta}q^\beta i\gamma_5 u$ is examined under G parity or under charge conjugation, it transforms with an opposite sign from $\gamma_\alpha i\gamma_5$. This means that $\gamma_\alpha i\gamma_5$ and $\sigma_{\alpha\beta}q^\beta i\gamma_5$ belong to different currents, for example, different isospin currents. If it is assumed that the weak interaction must belong to a definite current under G or C or CP, then the F_A terms are excluded. There is no direct experimental evidence to illuminate this point.

The amplitude is now reduced to

$$\langle p_2 p_L |T| p_1 p_\nu \rangle = \frac{1}{\sqrt{2}}\bar{u}(p_2)\left[\gamma_\lambda(G_V + G_A i\gamma_5) + \sigma_{\lambda\beta}\frac{F_V}{M_1 + M_2}\right] u(p_1)\, \bar{u}(p_L)\, \gamma^\lambda(1 + i\gamma_5)\, u(p_\nu),$$
$$= \frac{1}{\sqrt{2}}\bar{u}(p_2)\left[\gamma_\lambda\Big((G_V + F_V) + G_A i\gamma_5\Big) - F_V\frac{(p_1 + p_2)_\lambda}{M_1 + M_2}\right] u(p_1)\, \bar{u}(p_L)\, \gamma^\lambda(1 + i\gamma_5)\, u(p_\nu), \tag{29}$$

using the reduction trick for $\sigma_{\lambda\beta}q^\beta$.

Next we evaluate $\overline{|<|T|>|^2}$. It is convenient to do this by computing first the lepton and baryon parts separately,

$$\overline{J_\alpha^{L^\dagger} J_\beta^L} = [(1 - i\gamma_5)\, \gamma_\alpha(\tilde{p}_L - m_L)\, \gamma_\beta(1 + i\gamma_5)\, \tilde{p}_\nu], \tag{30}$$

remembering that trace is implied.

Let us ignore m_L. Even for muons, this results in an error of a few percent in cross sections. Let us also use $M_1 = M_2 = M$. The nucleon mass difference is so small, this introduces negligible error. Then,

$$\left[(1 - i\gamma_5)\, \gamma_\alpha \tilde{p}_L \gamma_\beta \tilde{p}_\nu\right] = 4\left(p_{L\alpha} p_{\nu\beta} + p_{\nu\alpha} p_{L\beta} - g_{\alpha\beta} p_L \cdot p_\nu + i\epsilon_{\alpha\xi\beta\eta} p_L^\xi p_\nu^\eta\right).$$

For the baryon traces, we have the following independent terms:

$$\overline{J_\alpha^{B^\dagger} J_\beta^B}$$

$$\overline{VV^\dagger}:\ \frac{1}{4}\left[\gamma^\alpha(\tilde{p}_1 + M)\, \gamma^\beta(\tilde{p}_2 + M)\right] = \left(p_1^\alpha p_2^\beta + p_2^\alpha p_1^\beta - g^{\alpha\beta}(p_1 \cdot p_2 - M^2)\right);$$

$$\overline{AA^\dagger}:\ \frac{1}{4}\left[\gamma^\alpha i\gamma_5(\tilde{p}_1 + M)\, \gamma^\beta i\gamma_5(\tilde{p}_2 + M)\right] = \left(p_1^\alpha p_2^\beta + p_2^\alpha p_1^\beta - g^{\alpha\beta}(p_1 \cdot p_2 + M^2)\right);$$

$$\overline{VA^\dagger}:\ \frac{1}{4}\left[\gamma^\alpha(\tilde{p}_1 + M)\, \gamma^\beta i\gamma_5(\tilde{p}_2 + M)\right] = + i\epsilon_{\alpha\xi\beta\eta} p_1^\xi p_2^\eta;$$

$$\overline{j'_\alpha j'_\beta}^{\dagger}:\ \frac{1}{4}\left[(\tilde{p}_1 + M)(\tilde{p}_2 + M)\right] = (p_1 \cdot p_2 + M^2);$$

$$\overline{V_\alpha j'_\beta}^{\dagger}:\ \frac{1}{4}\left[\gamma^\alpha(\tilde{p}_1 + M)(\tilde{p}_2 + M)\right] = M(p_1 + p_2)^\alpha;$$

$$\overline{A_\alpha j'_\beta}^{\dagger}:\ \frac{1}{4}\left[\gamma^\alpha i\gamma_5(\tilde{p}_1 + M)(\tilde{p}_2 + M)\right] = 0.$$

Simplification results from introducing

$$j_\lambda = p_{L\lambda} + p_{\nu\lambda},$$

$$q_\lambda = p_{1\lambda} - p_{2\lambda} = p_{L\lambda} - p_{\nu\lambda},$$

$$j'_\lambda = p_{1\lambda} + p_{2\lambda},$$

and

$$q^2 \equiv t = (p_1 - p_2)^2 = (p_L - p_\nu)^2.$$

Then,

$$p_1 = \frac{1}{2}(j' + q),$$

$$p_2 = \frac{1}{2}(j' - q),$$

$$p_L = \frac{1}{2}(j + q),$$

and

$$p_\nu = \frac{1}{2}(j - q).$$

Note that

$$q \cdot j = p_L^2 - p_\nu^2 = m_L^2 \approx 0;$$

$$q \cdot j' = p_1^2 - p_2^2 \approx 0;$$

$$p_1^\alpha p_2^\beta + p_2^\alpha p_1^\beta = \frac{1}{4}\Big((j' + q)^\alpha (j' - q)^\beta + (j' - q)^\alpha (j' + q)^\beta\Big) = \frac{1}{2}(j'^\alpha j'^\beta - q^\alpha q^\beta).$$

Also,

$$p_1 \cdot p_2 = M^2 - t/2,$$

and

$$p_L \cdot p_\nu \approx - t/2.$$

Thus,

$$\overline{J_\alpha^{L^\dagger} J_\beta^L} = 2\Big(j_\alpha j_\beta - q_\alpha q_\beta + g_{\alpha\beta} t + i\epsilon_{\alpha\beta\xi\eta} j^\xi q^\eta\Big),$$

and

$$\overline{VV^\dagger}: \ \frac{1}{2}(j'^\alpha j'^\beta - q^\alpha q^\beta + g^{\alpha\beta} t);$$

$$\overline{AA^\dagger}: \ \frac{1}{2}\Big(j'^\alpha j'^\beta - q^\alpha q^\beta - g^{\alpha\beta}(4M^2 - t)\Big);$$

$$\overline{VA^\dagger}: \ \frac{1}{2}\epsilon_{\alpha\beta\xi\eta} j'^\xi q^\eta;$$

$$\overline{j'j'^\dagger}: \ \frac{1}{2}(4M^2 - t);$$

$\overline{V^{\alpha}{}_{j'}{}^{,\beta^{\dagger}}}$: Mj'^{α};

$\overline{A^{\alpha}{}_{j'}{}^{,\beta^{\dagger}}}$: 0.

Finally, we construct $\overline{|<|T|>|^2}$ by summing the results from the table of $VV^{\dagger}$, etc., and simplifying the expressions,

$$\overline{|<|T|>|^2} = 2\left\{\left[G_V^2 + G_A^2 - \frac{F_V^2 t}{4M^2}\right]\left[(j \cdot j')^2 + t(4M^2 - t)\right]\right.$$
$$\left. + (G_\alpha + F_V)^2\, 2t^2 - G_A^2\, 2t(4M^2 - t) \pm (G_V + F_V)\, G_A t j \cdot j'\right\}, \tag{31}$$

where $t \equiv q^2$ is one of the invariant variables, and G_V, G_A, and F_V are all functions of t. We can work out the values of $j \cdot j'$ in terms of the invariant variables t and $s = (p_\nu + p_1)^2 = (p_2 + p_L)^2$. In terms of laboratory variables, in the limit, $m_L \approx 0$. Then $t = (p_\nu - p_L)^2 = -2E_V E_L (1 - \cos\theta_L) = (p_1 - p_2)^2 = 2M(M - E_2)$, and $s = M^2 + 2ME_\nu$.

To evaluate $j \cdot j' = (p_L + p_\nu) \cdot (p_1 + p_2)$, we use the following relations:

$p_L + p_\nu = q + 2p_\nu$;

$q \cdot j' \approx 0$;

$p_1 + p_2 = 2p_1 - q$;

$q \cdot p_\nu = -q^2/2$;

$p_\nu \cdot p_1 = \frac{1}{2}(s - M^2)$.

The result is $j \cdot j' = 2(s - M^2) + t$.

For neutrino experiments, the flux factor for the laboratory frame is

$$f = 2E_\nu 2M_1. \tag{32}$$

Since the relative velocity is $c = 1$, M_1 is the mass of the target nucleon.

The phase space appropriate for neutrino experiments is a laboratory two-body phase space,

$$\int d\rho_2 = \int \delta^{(4)}(p_i - p_L - p_2)\, \delta(p_L^2 - m_L^2)\, \delta(p_2^2 - M_2^2)\, \frac{d^4p_L d^4p_2}{(2\pi)^2}, \tag{33}$$

where p_i is the total momentum of the incoming system. We can eliminate the constraints due to the δ functions one at a time.

Integrate over d^4p_2

$$\int d\rho_2 = \int \delta(p_L^2 - m_L^2)\, \delta\Big((p_i - p_L)^2 - M_2^2\Big) \frac{d^4p_L}{(2\pi)^2}.$$

Write $p_L = (E_L, \underset{\sim}{P}_L)$, $p_i = (E_i, \underset{\sim}{P}_i)$, and $d^4p_L = dE_L P_L^2 dP_L d\Omega_L$. Integration over dE_L gives $(2E_L)^{-1}$ along with $E_L = \sqrt{P_L^2 + m_L^2}$, the root from $\delta(E_L^2 - P_L^2 - m_L^2)$.[7] This leaves

$$\int \delta\left[\left(E_i - \sqrt{P_L^2 + m_L^2}\right)^2 - (\underset{\sim}{P}_i - \underset{\sim}{P}_L)^2 - M_2^2\right] \frac{P_L^2 dP_L d\Omega_L}{(2\pi)^2\, 2\sqrt{P_L^2 + m_L^2}}.$$

Next, do we want a cross-section differential with respect to dP_L or $d\Omega_L$? First let us choose $d\Omega_L$; this means eliminating P_L. Call $\underset{\sim}{P}_i \cdot \underset{\sim}{P}_L = P_iP_L \cos\theta_L$. We write the above argument of the δ function in a mixed three- and four-dimensional form $\delta\Big(p_i^2 + m_L^2 - 2(E_i\sqrt{P_L^2 + m_L^2} - P_iP_L\cos\theta_L) - M_2^2\Big)$, and take the derivative to obtain

$$d\rho_2 = \frac{P_L^2 d\Omega_L}{(4\pi)^2(E_iP_L - P_iE_L\cos\theta_L)}, \tag{34}$$

where $E_L = \sqrt{P_L^2 + m_L^2}$, and P_L is the fairly complicated function of angle obtained by solving for P_L in the above constraint; i.e., it is determined by the kinematics, once the angle is given.

If we were interested in $d\sigma/dP_L$ instead, then the integration would have been over $d\cos\theta_L$. A subsequent integration over $d\phi_L$ then gives 2π. Performing the ϕ integration is only possible if it is known that the transition probability is independent of ϕ_L. The result is

$$d\rho_2 = \frac{P_L dP_L}{8\pi P_iE_L}. \tag{35}$$

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[7] We have used the following formula here:

$$\boxed{\delta[f(x)] = \frac{1}{|f'(x_0)|}\,\delta(x - x_0)}$$

where x_0 is a zero of $f(x)$, $f(x_0) = 0$. To see this, expand $f(x)$ inside the δ function $\delta\big(\cancel{f(x_0)} + f'(x_0)(x - x_0) + \dots\big)$, and now use rule $\boxed{\delta(ax) = \frac{1}{|a|}\,\delta(x)}$, which comes from $\int \delta(ax)dx = \int \delta(ax)[d(ax)]/a$.

Again, $E_L = \sqrt{P_L^2 + m_L^2}$, and $\cos\theta$ is a function of P_L given by the kinematical constraint.

If we are interested in the baryon angle or momentum, we need only interchange P_L and P_2 in the phase space formulas.

We now have all the ingredients to obtain the differential cross section,

$$d\sigma = \frac{1}{f} \overline{|<|T|>|^2}\, d\rho_2, \tag{36}$$

where f, $d\rho_2$, and $\overline{|<|T|>|^2}$ are given by Eqs. (32), (34) and (35), and (31), respectively.

II. Hyperon Production in an SU_3 Model*

This work was a collaboration with N. Cabbibo, and is somewhat based on an earlier letter by Cabbibo in Physical Review Letters on an SU_3 model of weak interactions, and a paper by myself in "Nuovo Cimento" discussing hyperon production by neutrinos.

Our principle motivation was to try to construct a definite test of the Cabibbo model in neutrino interactions. Such a test would depend on the quantitative details. Some numerical results are shown at the end.

The $\Delta S = \Delta Q$ rule permits only three hyperon interactions:

$$\overline{\nu} + p \rightarrow \Lambda + \mu^+,$$

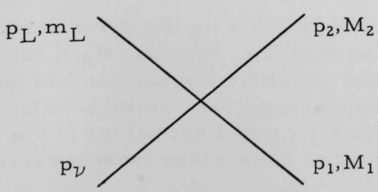
$$\overline{\nu} + n \rightarrow \Sigma^- + \mu^+,$$

and

$$\overline{\nu} + p \rightarrow \Sigma^0 + \mu^+. \tag{37}$$

The third reaction is related to the second by the $\Delta I = 1/2$ rule,

$$d\sigma(\Sigma^0) = \frac{1}{2} d\sigma(\Sigma^-).$$



*This lecture was given at a Theoretical Seminar.

The general amplitude for $\overline{\nu} + N \to B + L$ has a current-current form. The lepton current is

$$J_\lambda^L = \overline{v}(\nu_\mu)\,\gamma_\lambda(1 + i\gamma_5)\,v(\mu^\dagger),$$

while the most general baryon current is

$$\langle B_2\,|J_\lambda^\dagger|\,B_1\rangle = \frac{1}{2}\overline{u}(p_2)\left\{\gamma_\lambda(G_V + G_A i\gamma_5) + \sigma_{\lambda\beta}q^\beta\left(\frac{F_V}{M_1 + M_2} - \frac{F_A}{M_2 - M_1}i\gamma_5\right)\right.$$

$$\left. + q_\lambda\left(-\frac{H_V}{M_1 + M_2} + \frac{H_A}{M_2 - M_1}i\gamma_5\right)\right\}u(p_1). \qquad (38)$$

The form factors are functions of the invariant momentum transfer, q^2, and are all real as a consequence of T or CP invariance.

The basic idea of the SU_3 model of Cabibbo is to divide the baryon current in $\Delta S = 0$ and $\Delta S = 1$ parts,

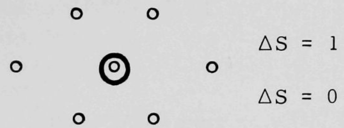
$$J_\lambda = \cos\theta\, J_\lambda^{(0)} + \sin\theta\, J_\lambda^{(1)}. \qquad (39)$$

That they should not have equal constants is already forced upon you by the experimental difference between the $\Delta S = 0$ and $\Delta S = 1$ rates. SU_3 enters through the assumption that the current transforms like an octet of SU_3. The angle θ can be determined from experiment. It is especially interesting that $\theta \approx 0.26$ follows from both K_{e3} to π_{e3} and the $K_{\mu 2}$ to $\pi_{\mu 2}$ decay-rate ratios. This is a nontrivial result since Clebsch-Gordan coefficients of SU_3 have been used. Since the strong interactions are different for all these decays, it suggests universality to some degree.

The most general expression for the baryon current can be simplified considerably. While H_A can possibly be related to G_A through the notion of an almost conserved current, on practical grounds this would be irrelevant since the H terms correspond to q_λ. The derivative q_λ acting on the lepton current gives m_L, and one can easily show that all terms in H^2 or H in the cross section have a factor m_L^2 in front of them. They are also typically about 1% in their contribution to the cross section and can be ignored.

For beta decay, Weinberg has shown that the F_A term has opposite character under the G transformation than G_A. He calls F_A a "second-class" current. This means that if the primitive Lagrangian had only G_A present, then F_A could not be induced by strong interactions. Also, if G_A is a member of an isospin current, then F_A could not belong to the same isospin current; it would have to belong to some other isospin current.

This kind of argument can be generalized to arbitrary multiplets, in our case octets. We look at an octet of currents -



Because the currents of interest to us are members of a multiplet, we can look at the transformation properties of other members of the multiplet under some transformation, say, charge conjugation. The neutral members for $\sigma_{\alpha\beta}q^{\beta}i\gamma_5$ transform with the opposite sign from $\gamma_\alpha i\gamma_5$. Therefore, they cannot be members of the same multiplet, and F_A could not be induced by SU_3 invariant strong interactions from G_A.

We call $\sigma_{\alpha\beta}q^{\beta}i\gamma_5$ a "second-class" current, and set F_A to zero in what follows. Note that q_λ is second class also, and H_V would have been set to zero if its contribution were not already ignorable on practical grounds. I see no reason why a nonzero F_A could not arise from symmetry-breaking interactions. Presumably, it would make a smaller contribution than the G_A term. There is no experimental evidence, either for or against these axial magnetism F_A terms.

The baryons belong to the 8 representation of SU_3. There are two 8's in $8 \otimes 8$; they are usually classified as odd and even under R, which is reflection through the origin in the weight diagram. In terms of two reduced matrix elements, using Clebsch-Gordan coefficients of SU_3,

$$\langle n |J_\lambda| \Sigma^- \rangle = \sin\theta\,(\mathcal{O}_\lambda - \mathcal{E}_\lambda);$$

$$\langle p |J_\lambda| \Lambda \rangle = -\sin\theta\sqrt{\frac{3}{2}}\left(\mathcal{O}_\lambda + \frac{1}{3}\mathcal{E}_\lambda\right). \tag{40}$$

For the vector parts, the conserved vector current hypothesis would have J^V_λ belonging to the same octet as the electromagnetic j_λ current, so we determine $\mathcal{O}_\lambda$ and $\mathcal{E}_\lambda$ from the electromagnetic current,

$$\langle p |j_\lambda| p \rangle = \mathcal{O}^{\nu}_{\lambda} + \frac{1}{3}\mathcal{E}^{\nu}_{\lambda};$$

$$\langle n |j_\lambda| n \rangle = -\frac{2}{3}\mathcal{E}^{\nu}_{\lambda}.$$

Then

$$\mathcal{O}^{\nu}_{\lambda} = \left[F_1^p(t) + \frac{1}{2}F_1^n(t)\right]\gamma_\lambda + \frac{1}{2M_N}\left[F_2^p(t) + \frac{1}{2}F_2^n(t)\right]\sigma_{\lambda\beta}q^\beta,$$

and

$$\mathcal{E}^{\nu}_{\lambda} = -\frac{3}{2}F_1^n(t)\,\gamma_\lambda - \frac{3}{4M_N}F_2^n(t)\,\sigma_{\lambda\beta}q^\beta, \tag{41}$$

where

$$F_1^p(0) = 1,\ F_1^u(0) = 0,\ F_2^p(0) = \mu_p, \text{ and } F_2^u(0) = \mu_n.$$

To determine the axial vector parts, we use

$$\langle p\,|J_\lambda|\,n\rangle = \cos\theta\,(\mathcal{O}_\lambda + \mathcal{E}_\lambda).$$

Now,

$$G_A\,|_{n\to p} = \cos\theta\left(G_A^{\mathcal{O}}(t) + G_A^{\mathcal{E}}(t)\right) = \cos\theta\,\tilde{G}_A(t),$$

$$\tilde{G}_A(0) = 1.25, \tag{42}$$

and a parametrization,

$$G_A^{\mathcal{O}}(t) = \tilde{G}_A(t)\,x\,(t),$$

$$G_A^{\mathcal{E}}(t) = \tilde{G}_A(t)\left(1 - x(t)\right). \tag{43}$$

From the decay of Σ^-, $x \approx 0.25$. It is very interesting that this 3-to-1, d-to-f ratio is the same as found in studies of strong interactions.

Finally,

$$\langle p\,|J_\lambda|\,\Lambda\rangle = -\frac{G}{\sqrt{2}}\sin\theta\sqrt{\frac{3}{2}}\left\{F_1^p(t)\,\gamma_\lambda + F_2^p(t)\,\frac{\sigma_{\lambda\beta}q^\beta}{2M_N} + \frac{1 + 2\,x\,(t)}{3}\,G_A(t)\gamma_\lambda i\gamma_5\right\};$$

and

$$\langle n\,|J_\lambda|\,\Sigma^-\rangle = \frac{G}{\sqrt{2}}\sin\theta\left\{\left[F_1^p(t) + 2F_1^n(t)\right]\gamma_\lambda + \frac{1}{2M_N}\left[F_2^p(t) + 2F_2^n(t)\right] \cdot \sigma_{\lambda\beta}q^\beta - \left(1 - 2\,x\,(t)\right)\tilde{G}_A(t)\,\gamma_\lambda i\gamma_5\right\}. \tag{44}$$

These expressions for the weak currents apply in the exact symmetry unit. How can symmetry breaking be included in the amplitude? Certain obvious corrections can be made although not entirely without ambiguity. We can replace $2M_N$ by $M_1 + M_2$. The isovector form factor is dominated by the ρ meson. Thus it seems reasonable to expect the K^* to dominate the strangeness changing form factors. This is consistent with present data on K_{e3} decay. Thus we have used

$$f(t) = \frac{M_{K^*}^2}{M_{K^*}^2 - t} \tag{45}$$

as the functional dependence of the form factors in our numerical examples.

Any possible dependence of $x(t)$ on t is too fine a detail to be experimentally evident for some time to come. We treat it as a constant. Virtually nothing is known about the functional dependence of $\tilde{G}_A(t)$, nor are any axial vector resonances known. Presumably it falls off also. For the purposes of numerical examples, we choose $f(t)$ for $\tilde{G}_A$ also.

Let us summarize our results.

For $\langle p\, |J_\lambda|\, \Lambda\rangle$,

$$\begin{aligned} G_V &= -G \sin\theta \sqrt{\frac{3}{2}}\, f(t); \\ G_A &= -1.25\, G \sin\theta \sqrt{\frac{3}{2}}\, \frac{1+2x}{3}\, f(t); \\ F_V &= -G \sin\theta \sqrt{\frac{3}{2}}\, \mu_p f(t). \end{aligned} \tag{46}$$

For $\langle n\, |J_\lambda|\, \Sigma^-\rangle$,

$$\begin{aligned} G_V &= G \sin\theta\, f(t); \\ G_A &= 1.25\, G \sin\theta (1-2x)\, f(t); \\ F_V &= G \sin\theta (\mu_p + 2\mu_n)\, f(t). \end{aligned} \tag{47}$$

In each case,

$F_A = 0$, $H_V = 0$, and H_A is ignored.

The formula for the differential cross section is

$$\frac{d\sigma}{dt}(s,t) = \frac{1}{32\pi k_{Bi}^2 s}\left\{\left[G_V^2 + G_A^2 - \frac{F_V^2 t}{(M_1+M_2)^2}\right]\right.$$

$$\times\Big((j \cdot j')^2 - (M_2^2 - M_1^2)^2 - (m_L^2 - t)(M_1+M_2)^2 + (M_2 - M_1)^2 - t\Big)$$

$$+ 2\left[(G_V+F_V)^2\Big((M_2 - M_1)^2 - t\Big) + G_A^2\Big((M_1+M_2)^2 - t\Big)\right](m_L^2 - t)$$

$$\left. - 4(G_V+F_V)\, G_A\left[j \cdot j't + m_L^2(M_2^2 - M_1^2)\right]\right\}, \tag{48}$$

where

$$k_{Bi} = \frac{s - M_1^2}{2\sqrt{s}},$$

and

$$j \cdot j' = (p_1+p_2) \cdot (k_\nu + p_L) = 2s - M_1^2 - M_2^2 - m_L^2 + t.$$

Expressions for the various laboratory $d\sigma/d\Omega$ or $d\sigma/dE$ are directly obtainable from $d\sigma/dt$; in fact, $d\sigma/dE_2 = 2M_1 d\sigma/dt$.

I would like to show some graphs of the results. I have a 7090 computer program that evaluates and plots the differential and total cross sections. Figures 1 and 2 are $d\sigma/d\cos\theta$ in both the lepton and baryon angles for Λ and Σ, and $E_\nu = 2$ BeV. The baryon cross section is plotted in a double-valued fashion. This corresponds to the baryon being forward in the laboratory for both the forward and backward leptons in the center of mass.

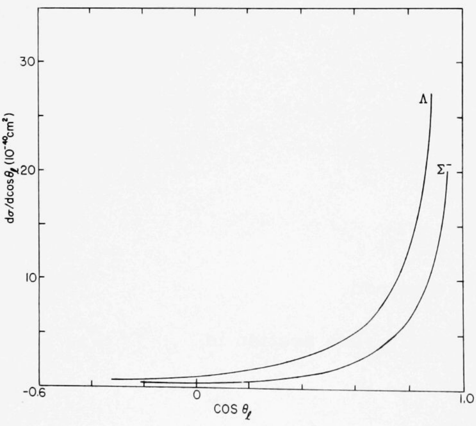


Fig. 1

Differential Cross Section as a Function of Lepton Angle

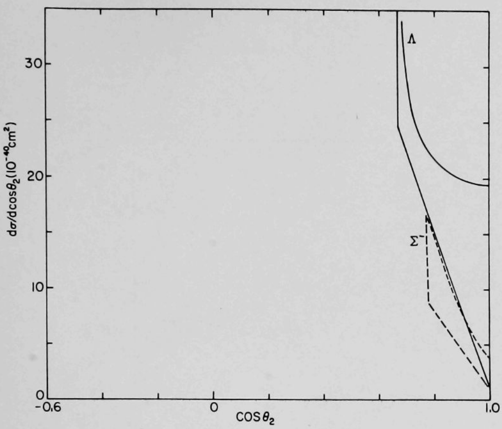


Fig. 2

Differential Cross Section as a Function of Baryon Angle

Notice the accummulation at the maximum angle. Figure 3 shows $d\sigma/dt$ for Λ and Σ and E_ν = 2 BeV, and Figure 4 shows the total cross sections. In each case, there are significant quantitative differences between the cross sections which could test the relative features of the model. We also predict the absolute magnitudes. It is also of interest to compare magnitudes of the total cross section with the nucleon cross sections; at 1 BeV, the ratio of $\overline{\nu} \rightarrow \Lambda$ to $\overline{\nu} \rightarrow n$ is about 1/15, while at 3 BeV the ratio is about 1/10.

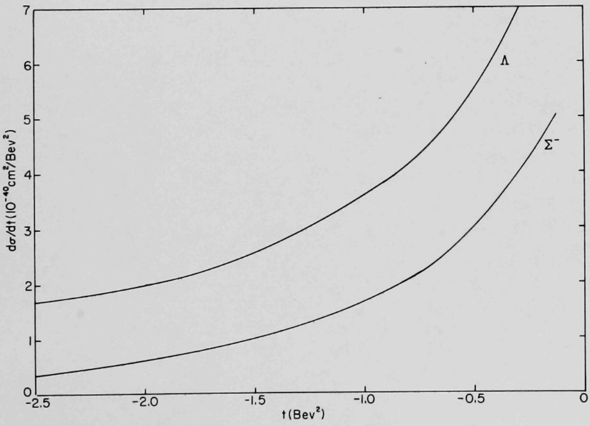


Fig. 3. Differential Cross Section as a Function of t

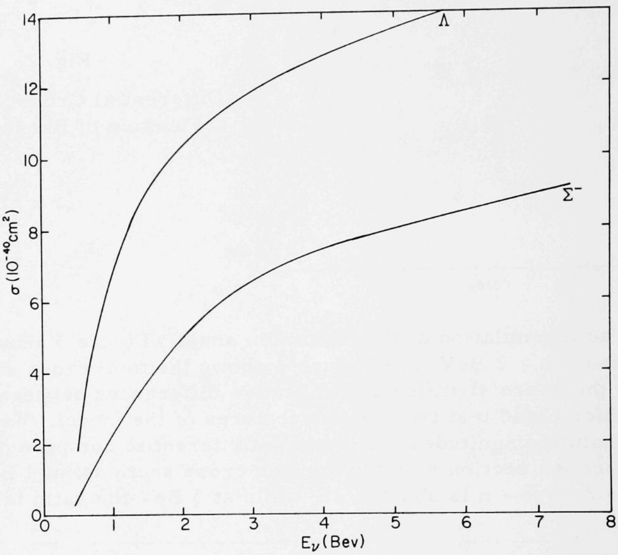


Fig. 4. Total Cross Section