January 15, 1971 FRA-TM-5

THE EFFECT OF ANISOTROPIC SCATTERING UPON THE ELASTIC MODERATION OF FAST NEUTRONS

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FRA TECHNICAL MEMORANDUM NO. 5

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Work performed under the auspices of the U. S. Atomic Energy Commision.

The Effect of Anisotropic Scattering Upon the Elastic Moderation of Fast Neutrons

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ABSTRACT

The effect of anisotropic (CM) elastic scattering upon neutron moderation has been investigated. Relations between the Legendre moments of the angular scattering data (CM) and the moderating parameters of the Greuling-Goertzel and improved Greuling-Goertzel theories are given, and it is shown theoretically and numerically that only the first few Legendre moments contribute significantly to the moderating parameters of the P_0 (lab) theory. Anisotropic scattering is shown, in general, to dramatically reduce the moderating effectiveness of a material, and to produce oscillations (in energy) in the neutron flux. Numerical examples for isotopes and mixtures representative of postulated fast-breeder reactors are given to quantitatively illustrate the effect of anisotropic scattering.

INTRODUCTION

The energy lost by a neutron in an elastic collision with an atomic nucleus is correlated with the angle through which the neutron scatters by the laws of classical mechanics. Any angular bias in the neutron scattering properties of a particular isotope is thus reflected in the parameters which characterize the moderation of neutrons by that isotope. If, for a given incident neutron energy, the scattering is predominantly forward in the center-of-mass system, then the isotope in question is a less effective moderator than it would be if the scattering distribution was isotropic in the center-of-mass system, and conversely for predominantly backward scattering.

The approximate slowing-down theory used in this paper has evolved from the original theory presented by Greuling, Clark, and Goertzel[1] for isotropic scattering. Hurwitz and Zweifel[2,3] introduced a formalism for treating anisotropic scattering. Amster[4] employed this formalism in extending the original theory to accommodate anisotropic scattering and in determining the sensitivity of the approximation to various order Legendre components of the angular scattering data. Subsequently, Greuling and Goertzel[5] extended their original work to accommodate anisotropic scattering. More recently, Stacey[6,7] developed a version of the theory which improved the treatment of wide and intermediate scattering resonances in mixtures. In both the basic G-G theory, given by Amster,[4] and generalized herein to explicitly define composite moderating parameters, and in the improved G-G theory given by Stacey[7] the contribution of anisotropic scattering to the moderating parameters can be represented explicitly. The theory is reviewed in an appendix.

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The appropriate slowing-down theory used in this paper has evolved from the original theory presented by Greuling, Clark, and Goertzel[1] for isotropic scattering. Murata and Meitel[2,3] introduced a formalism for treating anisotropic scattering. Amster[4] employed this formalism in extending the original theory to accommodate anisotropic scattering and in determining the sensitivity of the equations to various order legendre components of the angular scattering data. Subsequently, Greuling and Goertzel[5] extended their original work to accommodate anisotropic scattering. More recently, Stacey[6,7] developed a version of the theory which improved the treatment of wide and intermediate scattering resonances in mixtures. In both the basic G-G theory, given by Amster,[4] and generalized herein to explicitly define composite moderating parameters, and in the improved G-G theory given by Stacey[7] the contribution of anisotropic scattering to the moderating parameters can be represented explicitly. The theory is reviewed in an appendix.

THEORY

The ℓ-*th* order Legendre component of the elastic slowing-down density is defined

$$q_\ell(u) \equiv \sum_i \int_{u-\beta_i}^{u} du' \int_{u}^{u'+\beta_i} du'' \, \Sigma_{s_i}^{\ell}(u' \to u'')\phi_\ell(u') , \qquad (1)$$

where ϕ_ℓ is the ℓ-*th* order Legendre component of the angular neutron flux and $\Sigma_{s_i}^{\ell}$ is the ℓ-*th* order Legendre component of the scattering transference function of isotope i in the L-system

$$\Sigma_{s_i}^{\ell}(u \to u') = \Sigma_{s_i}(u) P_\ell\left(\mu_L(u \to u')\right) \left[\sum_{\ell'=0}^{L} \frac{2\ell' + 1}{2} \left(\frac{\sigma_i^{\ell'}(u)}{\sigma_i^{0}(u)} \right) P_{\ell'}\left(\mu_C(u \to u')\right) \right]$$

$$\times \left[\frac{2e^{u-u'}}{1 - \alpha_i} \right] , \qquad (2)$$

with $\alpha_i \equiv \left(M_i - 1\right)^2 / \left(M_i + 1\right)^2$, $\beta_i \equiv -\ln \alpha_i$, and M_i being the atomic mass of isotope i. The quantities σ_i^{ℓ} are the Legendre components of the scattering cross section in the center-of-mass (CM) system (only values for $\ell \leq L$ are assumed to be available), μ_L and μ_C are cosines of the scattering angle in the L and CM systems, respectively, Σ_{s_i} is the macroscopic scattering cross section, and P_ℓ is the ℓ-*th* Legendre polynomial. The independent variable u is the neutron lethargy.

An approximate slowing-down theory is obtained by assuming a slowly varying collision density over the scattering interval[7] (see appendix)

$$F_\ell(u') \equiv \Sigma(u')\phi_\ell(u') \simeq F_\ell(u) + (u' - u) \frac{dF_\ell(u)}{du} , \quad u > u' > u - \beta_i \qquad (3)$$

THEORY

The i-th order Legendre component of the elastic slowing-down density is defined

$$q_i(u) \equiv \sum_{i'} \int_{u-E_i}^{u} du' \int_{u'}^{u'+E_i} du'' \, \xi_i^{s}(u' \to u'')\phi_i(u'') \, , \qquad (1)$$

where ϕ_i is the i-th order Legendre component of the angular neutron flux and ξ_i^{s} is the i-th order Legendre component of the scattering transference function of isotope i in the L-system

$$\xi_i^{s}(u \to u') = \sum_{i=1}^{\infty} (u)P_i(\mu_0(u \to u')) \left[\sum_{k=0}^{\infty} \frac{2k+1}{2} \left[\frac{\sigma_k^{i}(u)}{\sigma_0^{i}(u)} \right] P_k(\mu_0(u \to u')) \right]$$

$$\times \left[\frac{2\alpha - 1 - \alpha^2}{1 - \alpha^2} \right] , \qquad (2)$$

with $\alpha_i = (M_i - 1)^2/(M_i + 1)^2$, $E_i = -\ln \alpha_i$, and M_i being the atomic mass of isotope i. The quantities σ_k^{i} are the Legendre components of the scattering cross section in the center-of-mass (CM) system tomic values for $k \leq L$ are assumed to be available), μ_0 and μ_0' are cosines of the scattering angle in the L and CM systems, respectively, ξ_i^{s} is the macroscopic scattering cross section, and P_k is the k-th Legendre polynomial. The independent variable u is the neutron lethargy.

An approximate slowing-down theory is obtained by assuming a slowly varying collision density over the scattering interval[7] (see above)

$$F_i(u') \equiv E(u')\phi_i(u') \cong F_i(u) + (u' - u) \frac{dF_i(u)}{du} \, , \quad u > u' > u - E_i \qquad (3)$$

in Eq. (1) and in the equation which results from differentiating Eq. (1) with respect to u[5] (Σ is the total cross section)

$$\frac{\bar{a}_\ell(u)}{e_\ell(u)} \frac{dq_\ell(u)}{du} = \hat{\xi}_\ell(u)\Sigma(u)\phi_\ell(u) - q_\ell(u) , \tag{4}$$

where

$$\hat{\xi}_\ell(u) = \bar{\xi}_\ell(u) + \frac{\bar{a}_\ell(u)c_\ell(u)}{e_\ell(u)} , \tag{5}$$

and the basic moderating parameters are defined as sums of isotopic moderating parameters which are given by

$$\bar{\xi}^i_\ell(u) \equiv \int_{u-\beta_i}^{u} du' \frac{\Sigma_{s_i}(u')}{\Sigma(u')} \int_{u}^{u'+\beta_i} du'' P_\ell(\mu_L) \sum_{\ell'=0}^{L} \frac{2\ell' + 1}{2} \frac{\sigma_i^{\ell'}(u')}{\sigma_i^0(u')}$$

$$\times P_{\ell'}(\mu_C) \frac{2e^{u'-u''}}{1 - \alpha_i} , \tag{6}$$

$$\bar{a}^i_\ell(u) \equiv \int_{u-\beta_i}^{u} du' \frac{\Sigma_{s_i}(u')}{\Sigma(u')} (u' - u) \int_{u}^{u'+\beta_i} du'' P_\ell(\mu_L) \sum_{\ell'=0}^{L} \frac{2\ell' + 1}{2} \frac{\sigma_i^{\ell'}(u')}{\sigma_i^0(u')}$$

$$\times P_{\ell'}(\mu_C) \frac{2e^{u'-u''}}{1 - \alpha_i} , \tag{7}$$

in Eq. (1) and in the equation which results from differentiating Eq. (1) with respect to $u_{\bar{1}}^{(3)}$ (I is the total cross section)

$$\frac{\bar{g}^{i}(u)}{e^{i}(u)} \frac{dq^{i}(u)}{du} = \bar{\xi}^{i}(u)I(u)e^{i}(u) - q^{i}(u) \; , \tag{4}$$

where

$$\bar{\xi}^{i}(u) = \xi^{i}(u) + \frac{\bar{g}^{i}(u)c^{i}(u)}{e^{i}(u)} \; , \tag{5}$$

and the basic moderating parameters are defined as sums of isotropic moderating parameters which are given by

$$\bar{\xi}_{\ell}^{i}(u) = \int_{u-B^{i}}^{u} du' \frac{\bar{g}e^{i}(u')}{\Sigma(u')} \int_{u}^{u'+B^{i}} du'' \, p^{i}(u'') \sum_{\ell'=0}^{L} \frac{2\ell'+1}{2} \frac{c_{\ell'}^{i}(u')}{c_{0}^{i}(u')}$$

$$\times p^{i'}(u') \frac{2e_{u'-u''}}{1-\alpha^{i}} \; , \tag{6}$$

$$\bar{\xi}_{\ell}^{i}(u) = \int_{u-B^{i}}^{u} du' \frac{\bar{g}e^{i}(u')}{\Sigma(u')} (u'-u) \int_{u}^{u'+B^{i}} du'' \, p^{i}(u'') \sum_{\ell'=0}^{L} \frac{2\ell'+1}{2} \frac{c_{\ell'}^{i}(u')}{c_{0}^{i}(u')}$$

$$\times p^{i'}(u') \frac{2e_{u'-u''}}{1-\alpha^{i}} \; . \tag{7}$$

$$c_\ell^i(u) \equiv \frac{\Sigma_{s_i}(u)}{\Sigma(u)} \int_{u-\beta_i}^{u} du' \, P_\ell\left(\mu_L\right) \sum_{\ell'=0}^{L} \frac{2\ell'+1}{2} \frac{\sigma_i^{\ell'}(u)}{\sigma_i^0(u)} P_{\ell'}\left(\mu_C\right) \frac{2e^{u'-u}}{1-\alpha_i}$$

$$- \int_{u-\beta_i}^{u} du' \, \frac{\Sigma_{s_i}(u')}{\Sigma(u')} P_\ell\left(\mu_L\right) \sum_{\ell'=0}^{L} \frac{2\ell'+1}{2} \frac{\sigma_i^{\ell'}(u')}{\sigma_i^0(u')} P_{\ell'}\left(\mu_C\right) \frac{2e^{u'-u}}{1-\alpha_i} , \quad (8)$$

$$e_\ell(u) \equiv \int_{u-\beta_i}^{u} du' \, \frac{\Sigma_{s_i}(u')}{\Sigma(u')} (u'-u) \, P_\ell\left(\mu_L\right) \sum_{\ell'=0}^{L} \frac{2\ell'+1}{2} \frac{\sigma_i^{\ell'}(u')}{\sigma_i^0(u')}$$

$$\times P_{\ell'}\left(\mu_C\right) \frac{2e^{u'-u}}{1-\alpha_i} . \quad (9)$$

When it is permissible to replace the cross section ratios under the integrals in Eqs. (6) through (9) with effective values, the isotopic moderating parameters may be written in terms of the lethargy moments of the CM-to-L transfer functions[2,3]

$$\bar{\xi}_\ell^i(u) = - \sum_{\ell'=0}^{L} \left\langle \frac{\Sigma_{s_i}}{\Sigma} \frac{\sigma_i^{\ell'}}{\sigma_i^0} \right\rangle_u T_{\ell\ell',i}^1 , \quad (10)$$

$$\bar{a}_\ell^i(u) = - \sum_{\ell'=0}^{L} \left\langle \frac{\Sigma_{s_i}}{\Sigma} \frac{\sigma_i^{\ell'}}{\sigma_i^0} \right\rangle_u T_{\ell\ell',i}^2 , \quad (11)$$

$$c_{\ell}^{\ell}(u) = \frac{E_{\ell}^{\ell}(u)}{\varepsilon(u)} \int_{u-E_{\ell}^{\ell}}^{u} du' \, p_{\ell}^{\ell}\left[u'\right] \sum_{k'=0}^{L} \frac{2k'+1}{2} \frac{a_{k'}^{\ell}(u)}{a_{0}^{\ell}(u)} P_{k'}^{\ell}\left(\mu^{c}\right) \frac{2\mu^{c\,u'-u}}{1-\sigma^{\ell}}$$

$$- \int_{u-E_{\ell}^{\ell}}^{u} du' \frac{\Sigma_{a}^{\ell}(u')}{\varepsilon(u')} p_{\ell}\left(u'\right) \sum_{k'=0}^{L} \frac{2k'+1}{2} \frac{a_{k'}^{\ell}(u')}{a_{0}^{\ell}(u')} P_{k'}^{\ell}\left(\mu^{c}\right) \frac{2\mu^{c\,u'-u}}{1-\sigma^{\ell}} \; ; \tag{8}$$

$$s_{\ell}^{\ell}(u) = \int_{u-E_{\ell}^{\ell}}^{u} du' \frac{\Sigma_{s}^{\ell}(u')}{\varepsilon(u')} (u'-u) \, p_{\ell}\left[u'\right] \sum_{k'=0}^{L} \frac{2k'+1}{2} \frac{a_{k'}^{\ell}(u')}{a_{0}^{\ell}(u')}$$

$$\times P_{k'}^{\ell}\left(\mu^{c}\right) \frac{2\mu^{c\,u'-u}}{1-\sigma^{\ell}} \; . \tag{9}$$

When it is permissible to replace the cross section ratios under the integrals in Eqs. (6) through (9) with effective values, the isotropic moderating parameters may be written in terms of the latency moments of the CM-to-L transfer functions (2.3)

$$\xi_{\ell}^{\ell}(u) = - \sum_{k'=0}^{L} \left\langle \frac{\xi s_{\ell}^{\ell}}{\xi} \frac{a_{k'}^{\ell}}{a_{0}^{\ell}} \right\rangle_{u} T_{2k'+1}^{\ell} \; , \tag{10}$$

$$\xi_{\ell}^{\ell}(u) = - \sum_{k'=0}^{L} \left\langle \frac{\xi s_{\ell}^{\ell}}{\xi} \frac{a_{k'}^{\ell}}{a_{0}^{\ell}} \right\rangle_{u} T_{2k'+1}^{2} \; , \tag{11}$$

$$c_\ell^i(u) = -\sum_{\ell'=0}^{L}\left[\frac{\Sigma_{s_i}(u)}{\Sigma(u)}\frac{\sigma_i^{\ell'}(u)}{\sigma_i^0(u)} - \left\langle\frac{\Sigma_{s_i}}{\Sigma}\frac{\sigma_i^{\ell'}}{\sigma_i^0}\right\rangle_u\right]T^0_{\ell\ell',i} , \tag{12}$$

$$e_\ell^i(u) = \sum_{\ell'=0}^{L}\left\langle\frac{\Sigma_{s_i}}{\Sigma}\frac{\sigma_i^{\ell'}}{\sigma_i^0}\right\rangle_u T^1_{\ell\ell',i} . \tag{13}$$

A rigorous definition of $\langle\ \rangle_u$ may be obtained by equating terms corresponding to the same value of ℓ' between Eqs. (6) and (10), (7) and (11), (8) and (12), and (9) and (13). However, a simple average may suffice for heavy mass materials. The primary purpose of writing the moderating parameters in this form is to indicate the functional similarity to the Greuling-Goertzel parameters which will be subsequently derived [Eqs. (6) and (7)] and discussed.

The appropriate form of the Greuling-Goertzel approximation with anisotropic scattering can be obtained by assuming (see appendix)

$$\Sigma_{s_i}^{\ell'}(u')\phi_\ell(u') \cong \Sigma_{s_i}^{\ell'}(u)\phi_\ell(u) + (u' - u)\frac{d}{du}\left[\Sigma_{s_i}^{\ell'}(u)\phi_\ell(u)\right] , \quad u > u' > u - \beta_i \tag{14}$$

$$\Sigma_{s_i}^{\ell} = \frac{\sigma_i^\ell}{\sigma_i^0}\cdot\Sigma_{s_i} ,$$

instead of Eq. (3), in the expressions for q_ℓ [Eq. (1)] and dq_ℓ/du. The equation analogous to Eq. (4) which results is

$$-\frac{a_\ell(u)}{\xi_\ell(u)}\frac{dq_\ell(u)}{du} = \xi_\ell(u)\left[1 + \frac{d}{du}\left(\frac{a_\ell(u)}{\xi_\ell(u)}\right)\right]\Sigma\ (u)\phi_\ell(u) - q_\ell(u) , \tag{15}$$

where the basic moderating parameters are defined as sums of isotopic moderating parameters which are given by

$$\sigma_i^i(u) = \sum_{i'=0}^{\frac{D}{2}} \left[\frac{\chi a_i(u)}{\tau(u)} \frac{\sigma_i^{i'}(u)}{\sigma_i^i(u)} \left\langle \frac{\tau^{a_i} \sigma_i^{i'}}{\chi \sigma_i^0} \right\rangle_u \right] \tau^{\delta_{i'i}} , \tag{12}$$

$$\sigma_i^i(u) = \sum_{i'=0}^{\frac{D}{2}} \left\langle \frac{\tau^{a_i} \sigma_i^{i'}}{\chi \sigma_i^0} \right\rangle_u \tau^{\delta_{i'i}} . \tag{13}$$

A rigorous definition of $\langle \ \rangle_u$ may be obtained by equating terms corresponding to the same value of i' between Eqs. (6) and (10), (7) and (11), (8) and (12), and (9) and (13). However, a simple average may suffice for heavy mass materials. The primary purpose of writing the moderating parameters in this form is to indicate the functional similarity to the Greuling-Goertzel parameters which will be subsequently derived [Eqs. (8) and (7)] and discussed.

The appropriate form of the Greuling-Goertzel approximation with anisotropic scattering can be obtained by assuming (see appendix)

$$\xi_i^{\mathrm{a}}(u')\phi_i^{\mathrm{a}}(u') \simeq \xi_i^{\mathrm{a}}(u)\phi_i^{\mathrm{a}}(u) + (u' - u) \frac{d}{du}\left[\xi_i^{\mathrm{a}}(u)\phi_i^{\mathrm{a}}(u)\right] , \quad u > u' > u - \delta_i^{\mathrm{a}}$$

$$\xi_i^{\mathrm{a}} = \frac{\sigma_i^{\mathrm{a}}}{\sigma_i^{\mathrm{s}}} \cdot \xi_i^{\mathrm{s}} ,$$

instead of Eq. (3), in the expressions for q_i^{a} [Eq. (1)] and dq_i^{a}/du. The equation analogous to Eq. (4) which results is

$$\frac{a_i(u)}{\xi_i^{\mathrm{a}}(u)} \frac{dq_i^{\mathrm{a}}(u)}{du} = \xi_i^{\mathrm{a}}(u) \left[1 + \frac{d}{du} \left(\frac{a_i(u)}{\xi_i^{\mathrm{a}}(u)} \right) \right] (u)\phi_i^{\mathrm{a}}(u) - q_i^{\mathrm{a}}(u) , \tag{15}$$

where the basic moderating parameters are defined as sums of isotropic moderating parameters which are given by

$$\xi_\ell^i(u) \equiv -\frac{\Sigma_{s_i}(u)}{\Sigma_s(u)} \sum_{\ell'=0}^{L} T^1_{\ell\ell',i} \frac{\sigma_i^{\ell'}(u)}{\sigma_i^0(u)}, \tag{16}$$

$$a_\ell^i(u) \equiv -\frac{\Sigma_{s_i}(u)}{\Sigma_s(u)} \sum_{\ell'=0}^{L} T^2_{\ell\ell',i} \frac{\sigma_i^{\ell'}(u)}{\sigma_i^0(u)}, \tag{17}$$

with

$$\Sigma_s \equiv \sum_i \Sigma_{s_i} .$$

Equations (4) or (15) may be combined with the P_N or B_N equations to eliminate the moments of the neutron flux.[7] With the assumption that $dq_\ell/du = 0$, $\ell \geq 1$, the result for the P_N equations is

$$\frac{dq_0(B,u)}{du} = -\frac{\Sigma_{ne}(B,u)q_0(B,u) + \varepsilon(u)S_0(u)}{M(B,u)}, \tag{18}$$

$$\phi_0(B,u) = \frac{q_0(B,u) + \gamma_0(u)S_0(u)}{M(B,u)}, \tag{19}$$

where

$$\Sigma_{ne}(B,u) = \Sigma(u) - \Sigma_s(u) - \frac{\eta^2}{3H_{1,N}(B,u)}, \tag{20}$$

with $H_{1,N}$ given by the continued fraction expansion

$$c_i^1(u) = -\frac{\Sigma s_i(u)}{\Sigma_s(u)} \sum_{k=0}^{\infty} T_{k+1}^{i} \frac{c_k^0(u)}{\Sigma_s^0(u)} , \tag{16}$$

$$s_i^1(u) = -\frac{\Sigma s_i(u)}{\Sigma_s(u)} \sum_{k=0}^{\infty} T_{k+1}^{i} \frac{s_k^0(u)}{\Sigma_s^0(u)} , \tag{17}$$

with

$$\Sigma_s = \sum_i \Sigma_s^i .$$

Equations (4) or (15) may be combined with the P_N or S_N equations to eliminate the moments of the neutron flux.[7] With the assumption that $dq_i/du = 0$, $i \geq 1$, the result for the P_N equations is

$$\frac{dq_0(B,u)}{du} = -\frac{\Sigma_{me}(B,u)q_0(B,u) + s(u)S_0(u)}{M(B,u)} , \tag{18}$$

$$\phi_0(B,u) = \frac{q_0(B,u) + \nu_s(u)S_0(u)}{M(B,u)} , \tag{19}$$

where

$$\Sigma_{me}(B,u) = \Sigma(u) - \Sigma_s(u) - \frac{u^2}{3H^1_{1,N}(B,u)} , \tag{20}$$

with $H^1_{1,N}$ given by the continued fraction expansion

$$H_{\ell,N} = b_{\ell-1} + \cfrac{a_\ell}{b_\ell + \cfrac{a_{\ell+1}}{b_{\ell+1} + \cfrac{a_{\ell+2}}{b_{\ell+2} + \cdots}}} , \tag{21}$$

terminating with b_{N-1}, where

$$b_\ell \equiv \Sigma(u) - \Sigma_s^{\ell+1}(u) , \qquad 0 \leq \ell \leq N - 1$$

$$a_\ell \equiv -\frac{\ell + 1}{2\ell + 1} \frac{\ell + 1}{2(\ell + 1) + 1} \eta^2 , \qquad 1 \geq \ell \geq N - 1 , \tag{22}$$

and with $\eta \equiv iB$ where B is the buckling. The quantity $H_{1,N}$ is an extended transport cross section which accounts in an approximate way for transport effects through order N. When N = 1, $H_{1,1}$ is the conventional transport cross section. For the approximate theory of Eq. (4)

$$M(B,u) \equiv \hat{\xi}_0(u)\Sigma(u) + \gamma_0(u)\Sigma_{ne}(B,u) , \tag{23a}$$

$$\gamma_0(u) \equiv \bar{a}_0(u)/e_0(u) , \tag{24a}$$

$$\varepsilon(u) \equiv \hat{\xi}_0(u)\Sigma(u) . \tag{25a}$$

For the approximate theory of Eq. (15)

$$M(B,u) \equiv \xi_0(u)\left[1 + \frac{d}{du}\left(\frac{a_0(u)}{\xi_0(u)}\right)\right]\Sigma_s(u) + \gamma_0(u)\Sigma_{ne}(B,u) , \tag{23b}$$

$$H_{k,N} = b_{k+1} + \cfrac{a_{k+2}}{b_{k+2} + \cfrac{a_{k+3}}{b_{k+3} + \cfrac{a_{k+4}}{b_{k+4} + \cdots}}} \tag{21}$$

terminating with b_{N-1}, where

$$b_k = I(u) - I_0^{k+1}(u) \,, \qquad 0 \le k \le N - 1$$

$$a_k = -\frac{k+1}{2n+1}\,\frac{k+1}{2(k+1)+1}\,\eta_k^2 \,, \qquad 1 \le k \le N - 1 \,, \tag{22}$$

and with $n = 1\beta$ where β is the buckling. The quantity $H_{1,N}$ is an extended transport cross section which accounts in an approximate way for transport effects through order N. When $N = 1$, $H_{1,1}$ is the conventional transport cross section. For the approximate theory of Eq. (4)

$$M(\beta,u) = \xi_0(u)\Sigma(u) + \tau_0(u)\Sigma_{tr}(\beta,u) \,, \tag{23a}$$

$$\tau_0(u) = \bar{a}_0(u)/a_0(u) \,, \tag{24a}$$

$$\varepsilon(u) = \xi_0(u)I(u) \,. \tag{25a}$$

For the approximate theory of Eq. (15)

$$M(\beta,u) = \xi_0(u)\left[1 + \frac{d}{du}\left(\frac{a_0(u)}{\xi_0(u)}\right)\right]I_0(u) + \tau_0(u)\Sigma_{tr}(\beta,u) \,, \tag{23b}$$

$$\gamma_0(u) \equiv -a_0(u)/\xi_0(u) , \tag{24b}$$

$$\varepsilon(u) \equiv \xi_0(u)\left[1 + \frac{d}{du}\left(\frac{a_0(u)}{\xi_0(u)}\right)\right]\Sigma_s(u) . \tag{25b}$$

Heavy mass expansions have been developed [4,8] for the quantities $T^1_{0\ell}$ and $T^2_{0\ell}$ which enter into the definitions of ξ_0 and a_0 in the Greuling-Goertzel theory,

$$T^1_{0\ell,i} = (-1)^{\ell+1}(2\ell + 1) \sum_{p=\ell+\delta_{\ell 0}}^{\infty} \frac{(p!)^2\left(1 - \alpha_i\right)^p}{p(p - \ell)!(p + \ell + 1)!} , \tag{26}$$

$$T^2_{0\ell,i} = (-1)^{\ell}(2\ell + 1) \sum_{p=\ell+\delta_{\ell 1}+2\delta_{\ell 0}}^{\infty} \frac{(p!)^2\left(1 - \alpha_i\right)^p}{p(p - \ell)!(p + \ell + 1)!} \sum_{j=1}^{p-1} \frac{1}{j} . \tag{27}$$

The salient features of Eqs. (26) and (27) are that the leading term in $T^1_{0\ell,i}$ is of order $\left(1 - \alpha_i\right)$ for $\ell < 2$ and of order $\left(1 - \alpha_i\right)^\ell$ for $\ell \geq 2$, and the leading term in $T^2_{0\ell,i}$ is of order $\left(1 - \alpha_i\right)^2$ for $\ell < 3$ and of order $\left(1 - \alpha_i\right)^\ell$ for $\ell \geq 3$. Making use of Eqs. (26) and (27), Eqs. (16) and (17) may be written

$$\xi_0 = \frac{\Sigma_{s_i}}{\Sigma_s}\left[A^i_1\left(\frac{\sigma^1}{\sigma^0}\right)\left(1 - \alpha\right) + \cdots + A_\ell\left(\frac{\sigma^1}{\sigma^0}, \cdots, \frac{\sigma^\ell}{\sigma^0}\right)\left(1 - \alpha\right)^\ell + \cdots\right] , \tag{28}$$

$$\gamma_0(u) = -a_0(u)/\varepsilon_0(u) \; , \tag{25a}$$

$$\alpha(u) = \varepsilon_0(u)\left[1 + \frac{d}{du}\left[\frac{a_0(u)}{\varepsilon_0(u)}\right]\right] \chi_0(u) \; . \tag{25b}$$

Heavy mass expansions have been developed [(4.8)] for the quantities $T^1_{0\alpha}$ and $T^2_{0\alpha}$ which enter into the definitions of ε_0 and a_0 in the Generalized theory.

$$T^1_{0\alpha,1} = (-1)^{\alpha+1}(2\alpha + 1) \sum_{p=\alpha+\alpha_{10}}^{\infty} \frac{(p!)^2 (1 - a^1)^p}{p(p - \alpha)!(p + \alpha + 1)!} \; , \tag{26}$$

$$T^2_{0\alpha,1} = (-1)^{\alpha}(2\alpha + 1) \sum_{p=\alpha+\alpha_{11}+2\alpha_{10}}^{\infty} \frac{(p!)^2 (1 - a^1)^p}{p(p - \alpha)!(p + \alpha + 1)!} \sum_{j=1}^{p-1} \frac{1}{j} \; . \tag{27}$$

The salient features of Eqs. (26) and (27) are that the leading term in $T^1_{0\alpha,1}$ is of order $(1 - a^1)$ for $\alpha = 2$ and of order $(1 - a^1)^\alpha$ for $\alpha \geq 2$, and the leading term in $T^2_{0\alpha,1}$ is of order $(1 - a^1)^2$ for $\alpha = 3$ and of order $(1 - a^1)^\alpha$ for $\alpha \geq 3$. Making use of Eqs. (26) and (27), Eqs. (16) and (17) may be written

$$\varepsilon_0 = \frac{1 - a^1}{\chi_0}\left[A^1_1\left(\frac{\sigma_1}{\sigma_0}\right)(1 - a) + \cdots + A^\alpha\left(\frac{\sigma_1}{\sigma_0}\right)^\alpha \cdots \left(\frac{\sigma_\alpha}{\sigma_0}\right)(1 - a)^\alpha + \cdots\right] \; , \tag{28}$$

$$\gamma_0^i \equiv -\frac{a_0^i}{\xi_0^i} = \left[B_2\left(\frac{\sigma_i^1}{\sigma_i^0}, \frac{\sigma_i^2}{\sigma_i^0}\right)\left(1 - \alpha_i\right) + \cdots + B_\ell^i\left(\frac{\sigma_i^1}{\sigma_i^0}, \cdots, \frac{\sigma_i^\ell}{\sigma_i^0}\right) \right.$$

$$\left. \times \left(1 + \alpha_i\right)^{\ell-1} + \cdots \right], \tag{29}$$

where the A_ℓ^i and B_ℓ^i are functions of the indicated arguments.

Some inferences about the effect of anisotropic scattering upon the moderating parameters can be drawn from Eqs. (28) and (29). To evaluate ξ_0^i or γ_0^i to accuracy of order $\left(1 - \alpha_i\right)^\ell$, it is only necessary to retain terms through order ℓ or $\ell + 1$, respectively, in the Legendre expansion of the scattering transference function in the CM system. For heavy elements (e.g. uranium) $\left(1 - \alpha_i\right) \simeq 0.01$ and for light-to-intermediate elements $\left(1 - \alpha_i\right) \simeq 0.1$. Thus, the first few terms in the Legendre decomposition of the elastic scattering transference function should suffice for the numerical evaluation of the moderating parameters of the Greuling-Goertzel theory. Because $1 - \alpha_i$ is proportional to $1/M_i$, to first order, the argument given in the appendix for the Greuling-Goertzel approximation being accurate through second order in $1/M_i$ suggests that no added accuracy can be obtained by retaining terms with $\ell > 2$ in the evaluation of the moderating parameters.

Examination of Eqs. (10) through (13) indicates that the moderating parameters of the improved theory have approximately the same functional dependence upon the atomic mass and the Legendre scattering moments as do the corresponding parameters of the Greuling-Goertzel theory. This observation is augmented by the fact that the improved theory reduces identically to the Greuling-Goertzel theory for constant cross sections and constant angular distributions. Consequently, the first few Legendre

$$\gamma_0^i \equiv -\frac{\epsilon_0^i}{\epsilon_1^i} = \left[B_2^i \left(\frac{a_1^i}{a_0^i}, \frac{a_2^i}{a_0^i} \right) \left(1 - a^i \right) + \cdots + B_\ell^i \left(\frac{a_1^i}{a_0^i}, \cdots, \frac{a_\ell^i}{a_0^i} \right) \right.$$

$$\left. \times \left(1 + a^i \right)^{\ell-1} + \cdots \right] . \tag{25}$$

where the A_ℓ^i and B_ℓ^i are functions of the indicated arguments.

Some inferences about the effect of anisotropic scattering upon the moderating parameters can be drawn from Eqs. (24) and (25). To evaluate ϵ_0^i or γ_0^i to accuracy of order $\left(1 - a^i\right)^k$, it is only necessary to retain terms through order ℓ or $\ell + 1$, respectively, in the Legendre expansion of the scattering transference function in the CM system. For heavy elements (e.g. uranium) $\left(1 - a^i\right) \geq 0.01$ and for light-to-intermediate elements $\left(1 - a^i\right) \geq 0.1$. Thus, the first few terms in the Legendre decomposition of the elastic scattering transference function should suffice for the numerical evaluation of the moderating parameters of the Greuling-Goertzel theory. Because $1 - a^i$ is proportional to $1/M^i$, to first order, the argument given in the appendix for the Greuling-Goertzel approximation being accurate through second order in $1/M^i$ suggests that no added accuracy can be obtained by retaining terms with $\ell > 2$ in the evaluation of the moderating parameters.

Examination of Eqs. (10) through (12) indicates that the moderating parameters of the improved theory have approximately the same functional dependence upon the atomic mass and the Legendre scattering moments as do the corresponding parameters of the Greuling-Goertzel theory. This observation is augmented by the fact that the improved theory reduces identically to the Greuling-Goertzel theory for constant cross sections and constant angular distributions. Consequently, the first few Legendre

moments of the scattering distribution should suffice in the numerical evaluation of the moderating parameters of the improved theory, also, and the argument that no additional accuracy can be obtained by retaining terms with $\ell > 2$ is approximately applicable.

The need to retain only the first few Legendre moments of the scattering distribution in evaluating moderating parameters represents a distinct computational advantage for the Greuling-Goertzel type of approximation relative to the direct use of group transfer cross sections in ultrafine multigroup approximations where the group width is narrower than the maximum lethargy gain per collision for most or all of the isotopes in the mixture. For such problems it is frequently necessary to retain 10-20 Legendre moments to accurately evaluate the group transfer cross sections, which actually represent the probabilities for scattering to within a relatively small angular interval, and consequently require higher-order moments of the angular scattering distribution for an adequate representation. In the P_0 Greuling-Goertzel type approximation the moderating parameters represent low-order moments of the slowing-down kernel which are determined primarily by the first few Legendre moments of the scattering distribution. For group widths greater than the maximum lethargy gain per collision, the first few Legendre moments of the angular scattering data should suffice to evaluate the group transfer cross sections, also.

NUMERICAL RESULTS

Several computations have been performed in order to obtain a quantitative appreciation of the effect of anisotropic elastic scattering

moments of the scattering distribution should suffice in the practical evaluation of the moderating parameters of the improved theory, also, and the argument that no additional accuracy can be obtained by retaining terms with $\ell > 2$ is approximately applicable.

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NUMERICAL RESULTS

Several computations have been performed in order to obtain a qualitative appreciation of the effect of anisotropic elastic scattering

upon neutron moderation. Angular scattering distribution data were taken from ENDF/B. Examples relevant to fast-breeder reactors were chosen. The buckling was zero in these examples.

Isotopic Moderating Parameters

The isotopic moderating parameters

$$\tilde{\xi}_0^i(u) \equiv -\sum_{\ell'=0}^{L} T^1_{0\ell',i} \frac{\sigma_i^{\ell'}(u)}{\sigma_i^0(u)} \tag{30}$$

and

$$\tilde{\gamma}_0(u) \equiv -\frac{\sum_{\ell'=0}^{L} T^2_{0\ell',i}\left[\sigma_i^{\ell'}(u)/\sigma_i^0(u)\right]}{\sum_{\ell'=0}^{L} T^1_{0\ell',i}\left[\sigma_i^{\ell'}(u)/\sigma_i^0(u)\right]}, \tag{31}$$

which are related to the parameters of Eqs. (16) and (17), were evaluated for several isotopes.

The quantities $\tilde{\xi}_0^i$ and $\tilde{\gamma}_0^i$ are plotted for ^{16}O, ^{23}Na, and ^{239}Pu in Figs. 1-6 for three different orders of anisotropy, L = 0, 1, and 2. The functional relationship indicated by Eqs. (28) and (29) is apparent from these results, in that only σ^1/σ^0 has a large effect on $\tilde{\xi}$, while σ^1/σ^0 and σ^2/σ^0 have a large effect on $\tilde{\gamma}$. The values of $\tilde{\xi}$ for L > 2 differed insignificantly from the values for L = 2. It is interesting that $\tilde{\gamma} < 0$ for ^{239}Pu when the series in Eq. (31) are terminated with L = 1. This was also noted for ^{56}Fe and other heavier mass isotopes. In general, it can be shown, to first order in a heavy mass expansion, that the condition for $\tilde{\gamma} < 0$ is

$$\frac{3}{2}\frac{\sigma_i^1}{\sigma_i^0} > 1 + \frac{1}{2}\frac{\sigma_i^2}{\sigma_i^0}.$$

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$$\bar{\xi}_0^i(u) = -\sum_{\ell=0}^{L} T_\ell^i \, \frac{\sigma_\ell^i(u)}{\sigma_0^i(u)} \tag{30}$$

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which are related to the parameters of Eqs. (18) and (17), were evaluated for several isotopes.

The quantities $\bar{\xi}_0^i$ and $\bar{\gamma}_0^i$ are plotted for ^{16}O, ^{23}Na, and ^{238}U, in Figs. 1-6 for three different orders of anisotropy, $L = 0$, 1, and 2. The functional relationship indicated by Eqs. (28) and (29) is apparent from these results, in that only σ_1/σ_0 has a large effect on $\bar{\xi}$, while σ_1/σ_0 and σ_2/σ_0 have a large effect on $\bar{\gamma}$. The values of $\bar{\xi}$ for $L = 2$ differed insignificantly from the values for $L = 2$. It is interesting that $\bar{\gamma} < 0$ for ^{238}U when the series in Eq. (31) are terminated with $L = 1$. This was also noted for ^{56}Fe and other heavier mass isotopes. In general, it can be shown, to first order in a heavy mass expansion, that the condition for $\bar{\gamma} < 0$ is

$$\frac{1}{2}\frac{\sigma_1^i}{\sigma_0^i} > 1 + \frac{1}{2}\frac{\sigma_2^i}{\sigma_0^i} .$$

When the Legendre expansion is terminated at L = 1, this condition becomes $\sigma_i^1/\sigma_i^0 > 2/3$. Thus, while it seems that the Legendre expansion can be terminated at L = 2 without significantly affecting the evaluation of the moderating parameters, termination at $L \leq 1$ can obviously introduce large errors.

The quantities $\tilde{\xi}$ and $\tilde{\gamma}$ for ^{56}Fe and ^{12}C are shown in Figs. 7 and 8 for L = 5. The energy intervals in which $\tilde{\xi}_{anis} > \tilde{\xi}_{iso}$ correspond to intervals in which the scattering is predominantly backwards in the CM system, and conversely $\tilde{\xi}_{anis} < \tilde{\xi}_{iso}$ where the scattering is predominantly forward. The variation in energy of the angular scattering distribution produces a resonance-like structure in the moderating parameters. Deviations of $\tilde{\xi}$ and $\tilde{\gamma}$ from their isotropic values are more significant, naturally, for the higher energies (above ~100 keV) where the anisotropic components of the scattering distribution are appreciable. At lower energies $\tilde{\xi}$ and $\tilde{\gamma}$ approach their isotropic values. For the isotopes examined (^{12}C, ^{16}O, ^{23}Na, ^{56}Fe, ^{238}U, and ^{239}Pu), the sensitivity of the moderating parameters to the retention of additional Legendre moments of the angular scattering data was consistent with the predictions of Eqs. (28) and (29); i.e. addition of the ℓ'-*th* moment influenced the result by no more than order $(1 - \alpha)^{\ell}$ and $(1 - \alpha)^{\ell-1}$ for γ.

Moderating Parameters for a Pseudo-Mixture

In order to clearly illustrate the effect of the scattering anisotropy upon neutron moderation a mixture of fictitious isotopes with constant scattering cross sections and zero non-elastic cross sections was considered (mixture 1 in Table I, with $\sigma_{ne} = 0$ and $\sigma_p^{16} = 4.0$). In such a mixture, neutrons slowing down from a mono-energetic source have a constant

When the Legendre expansion is terminated at $L = 1$, this condition becomes $\sigma_1^1/\sigma_1^0 > 2/3$. Thus, while it seems that the Legendre expansion can be terminated at $L = 2$ without significantly affecting the evaluation of the moderating parameters, termination at $L \leq 1$ can obviously introduce large errors.

The quantities $\bar{\xi}$ and $\bar{\gamma}$ for ^{56}Fe and ^{12}C are shown in Figs. 7 and 8 for $L = 5$. The energy intervals in which $\bar{\xi}_{ana} > \bar{\xi}_{iso}$ correspond to intervals in which the scattering is predominantly backward in the CM system, and conversely $\bar{\xi}_{ana} < \bar{\xi}_{iso}$ where the scattering is predominantly forward. The variation in energy of the angular scattering distribution produces a resonance-like structure in the moderating parameters. Deviations of $\bar{\xi}$ and $\bar{\gamma}$ from their isotropic values are more significant, naturally, for the higher energies (above 400 keV) where the anisotropic components of the scattering distribution are appreciable. At lower energies $\bar{\xi}$ and $\bar{\gamma}$ approach their isotropic values. For the isotopes examined (^{12}C, ^{16}O, ^{23}Na, ^{56}Fe, ^{238}U, and ^{239}Pu), the sensitivity of the moderating parameters to the inclusion of additional Legendre moments of the angular scattering data was consistent with the predictions of Eqs. (28) and (29); i.e., addition of the ℓ-th moment influenced the result by no more than order $(1 - \alpha)^{\ell}$ and $(1 - \alpha)^{\ell-1}$ for γ.

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In order to clearly illustrate the effect of the scattering anisotropy upon neutron moderation a mixture of fictitious isotopes with constant scattering cross sections and zero non-elastic cross sections was considered (mixture 1 in Table 1, with $\sigma_{ne} = 0$ and $\sigma_{\gamma}^{i0} = 0.0$). In such a mixture, neutrons slowing down from a mono-energetic source have a constant

elastic slowing-down density, as may be inferred from Eq. (18) and from physical considerations. If the elastic scattering were isotropic the moderating parameters, and hence the flux given by Eq. (19), would be constant in lethargy.

The parameters ξ_0 of the Greuling-Goertzel theory [sum over isotopes of the ξ_0^i of Eq. (16)] and $\hat{\xi}_0$ of the improved theory [Eq. (5)] were computed for the pseudo-mixture. To evaluate $\hat{\xi}_0$, the isotopic parameters of Eqs. (6) through (9) were evaluated numerically with the recursion relations given in Ref. 7.[a] For ^{239}Pu, Eqs. (10) through (13) with the heavy mass expansions of Eqs. (26) and (27), and with an arithmetic average of endpoint values used to define effective cross section ratios, yielded practically identical results; however, such a simple average was not sufficient for the lighter mass isotopes. The moderating parameter ξ_0 of the Greuling-Goertzel theory reflects the lethargy dependence of the angular scattering distribution directly, whereas $\hat{\xi}_0$ of the improved theory involves integrals over the scattering lethargy interval of the angular scattering distribution. Consequently, $\hat{\xi}_0$ is a somewhat smoother function of lethargy than is ξ_0. The flux defined by Eqs. (19) and (23), which is inversely proportional to ξ_0 or $\hat{\xi}_0$ for this case,[b] is plotted for the two calculations in Fig. 9. The variation of the flux is due entirely to the variation in lethargy of the angular scattering distribution, and the effect of the latter in this regard is similar to that of wide scattering resonances.

Elastic Moderation in Fast-Reactor Mixtures

The elastic slowing-down density was computed in mixtures typical of those postulated for fast-breeder reactor cores and blankets. Compositions

elastic slowing-down density, as may be inferred from Eq. (15) and from physical considerations. If the elastic scattering were isotropic the moderating parameter, and hence the flux given by Eq. (15), would be constant in lethargy.

The parameters ξ_0 of the Greuling-Goertzel theory [sum over isotopes of the ξ_0^i of Eq. (16)] and ξ_0^* of the improved theory [Eq. (5)] were computed for the pseudo-mixture. To evaluate ξ_0^*, the isotopic parameters of Eqs. (6) through (9) were evaluated numerically with the resonance relations given in Ref. 7.[8] For ^{238}U, Eqs. (10) through (13) with the heavy mass expansions of Eqs. (26) and (27), and with an arithmetic average of endpoint values used to define effective cross section ratios, yielded practically identical results; however, such a simple average was not sufficient for the lighter mass isotopes. The moderating parameter ξ_0 of the Greuling-Goertzel theory reflects the lethargy dependence of the angular scattering distribution directly, whereas ξ_0^* of the improved theory involves integrals over the scattering lethargy interval of the angular scattering distribution. Consequently, ξ_0^* is a somewhat smaller function of lethargy than is ξ_0. The flux defined by Eqs. (18) and (23), which is inversely proportional to ξ_0 or ξ_0^* for this case, is plotted for the two calculations in Fig. 2. The variation of the flux is due entirely to the variation in lethargy of the angular scattering distribution, and the effect of the latter in this regard is similar to that of wide scattering resonances.

Elastic Moderation in Fast-Reactor Mixtures

The elastic slowing-down density was computed in mixtures typical of those postulated for fast-breeder reactor cores and blankets. Compositions

and "background" cross sections are given in Table I, and Table II contains the parameters which were used to represent the wide scattering resonances of oxygen and iron.

To isolate the anisotropic effects, the attenuation of the elastic slowing-down density due to non-elastic events was computed without including the buildup due to fission, inelastic scattering, and n - 2n events. Typical results[c] are depicted in Figs. 10-12. The notations IGG and GG refer to the improved Greuling-Goertzel [Eqs. (23a), (24a), and (25a)] and the standard Greuling-Goertzel [Eqs. (23b), (24b), and (25b)] prescriptions, respectively.

Inclusion of the anisotropic scattering effects in the evaluation of the moderating parameters, which reduces the parameter M of Eq. (23) relative to its isotropic value, in general, dramatically increases the attenuation of the elastic slowing-down density. This effect of anisotropic scattering is more pronounced at higher energies, where the anisotropy in the angular scattering data is greater, as may be seen by comparing Figs. 10 and 11.

The IGG parameters result in somewhat more attenuation than do the GG parameters. This is due, at least in part, to the failure of the GG parameters to predict sufficient flux peaking just below (in energy) strong scattering resonances, as discussed previously.[6]

Strong scattering resonances reduce the attenuation rate by increasing the ratio of elastic-to-nonelastic events. The "flattening-out" of the elastic slowing-down density in the vicinity of the 1.0-MeV ^{16}O resonance is evident in Figs. 10 and 12, and the effect of the 27.9 keV ^{56}Fe resonance is apparent in Fig. 11.

and "backward" cross sections are given in Table I, and Table II contains the parameters which were used to represent the wide scattering resonances of oxygen and iron.

To isolate the anisotropic effects, the attenuation of the elastic slowing-down density due to non-elastic events was computed without including the buildup due to fission, inelastic scattering, and n - 2n events. Typical results are depicted in Figs. 10-12. The notations IGE and GE refer to the improved Grueling-Goertzel [Eqs. (23a), (24a), and (25a)] and the standard Grueling-Goertzel [Eqs. (23b), (24b), and (25b)] prescriptions, respectively.

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The IGE parameters result in somewhat more attenuation than do the GE parameters. This is due, at least in part, to the failure of the GE parameters to predict sufficient flux peaking just below (in energy) strong scattering resonances, as discussed previously.[6]

Strong scattering resonances reduce the attenuation rate by increasing the ratio of elastic-to-nonelastic events. The "flattening-out" of the elastic slowing-down density in the vicinity of the 1.0-keV ^{56}O resonance is evident in Figs. 10 and 12, and the effect of the 27.9 keV ^{56}Fe resonance is apparent in Fig. 11.

CONCLUSIONS

The effects of elastic scattering anisotropy (in the CM) can be incorporated directly into the moderating parameters of the Greuling-Goertzel type theory. Only the first few Legendre moments of the angular scattering data need be retained to evaluate the moderating parameters for the P_0 (lab) slowing-down thoery, consistent with predictions based only upon atomic mass. The anisotropic moderating parameters are generally smaller than the same parameters evaluated for isotropic scattering only, implying less effective moderating ability for the former than the latter, which reflects the predominantly forward bias in the angular scattering data. Elastic slowing-down densities computed with the anisotropic moderating parameters are attenuated much more rapidly than when the moderating parameters are computed with only isotropic scattering. The rapid variation with energy of the angular scattering distribution results in a resonance-like structure in the moderating parameters which produces variations in the flux similar to those produced by wide scattering resonances. Significant differences are found in the moderating parameters, and consequently the elastic slowing-down density and flux, computed by the Greuling-Goertzel and by the improved Greuling-Goertzel prescriptions.

ACKNOWLEDGMENTS

The author is grateful to E. M. Pennington for assistance in reducing the ENDF/B angular scattering data to the format employed in the computations, and to J. P. Regis for assistance in developing computer plotting routines which facilitated evaluation of the results.

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The effects of elastic scattering anisotropy (in the CM) can be incorporated directly into the moderating parameters of the grouling-Goertzel type theory. Only the first few Legendre moments of the angular scattering data need be retained to evaluate the moderating parameters for the P_0 (lab) slowing-down theory, consistent with predictions based only upon atomic mass. The anisotropic moderating parameters are generally smaller than the same parameters evaluated for isotropic scattering only, implying less effective moderating ability for the former than the latter, which reflects the preferentially forward bias in the angular scattering data. Elastic slowing-down densities computed with the anisotropic moderating parameters are attenuated much more rapidly than when the moderating parameters are computed with only isotropic scattering. The rapid variation with energy of the angular scattering distribution results in a resonance-like structure in the moderating parameters which produces variations in the flux similar to those produced by wide scattering resonances. Significant differences are found in the moderating parameters, and consequently the elastic slowing-down density and flux, computed by the Grouling-Goertzel and by the improved Grouling-Goertzel presentations.

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APPENDIX

Review of Greuling-Goertzel Type Theory

Amster[4] and Fenziger and Zweifel[9] presented generalized derivations of the Greuling-Goertzel theory including anisotropic scattering. The derivation presented here is slightly different, being an extension of Williams'[10] derivation to the case of anisotropic scattering. The notation of Amster,[4] generalized to denote isotopic components of a mixture, will be used.

If Eq. (2) is substituted into Eq. (1) and a Taylor's series expansion is made in the integrand

$$\Sigma_{s_i}^{\ell'}(u')\phi_\ell(u') = \sum_{n=0}^{\infty} \frac{(u' - u)^n}{n!} \frac{d^n}{du^n} \left[\Sigma_{s_i}^{\ell'}(u)\phi_\ell(u)\right] , \tag{A.1}$$

where

$$\Sigma_{s_i}^{\ell'} \equiv \Sigma_{s_i} \frac{\sigma_i^{\ell'}}{\sigma_i^0} ,$$

Eq. (1) may be written

$$q_\ell(u) = - \sum_i \sum_{n=0}^{\infty} \frac{d^n}{du^n} \left[G_\ell^{n+1}(u)\phi_\ell(u)\right] , \tag{A.2}$$

where

APPENDIX

Review of Grenning-Goertzel Type Theory

Ahmed[4] and Fenlayer and Zweifel[5] presented generalized derivations of the Grenning-Goertzel theory including anisotropic scattering. The derivation presented here is slightly different, being an extension of Williams'[10] derivation to the case of anisotropic scattering. The notation of Ahmed,[4] generalized to denote isotopic components of a mixture, will be used.

If Eq. (2) is substituted into Eq. (1) and a Taylor's series expansion is made in the integrand

$$E_{k'}^{s^i}(u')\phi_{k'}^{s}(u') = \sum_{n=0}^{\infty} \frac{(u' - u)^n}{n!} \frac{d^n}{du^n} \left[E_{k'}^{s^i}(u)\phi_{k'}^{s}(u) \right] , \tag{A.1}$$

where

$$E_{k'}^{s^i} \equiv E^{s^i} \frac{\sigma_0^i}{\sigma_{k'}^i} ,$$

Eq. (1) may be written

$$d^s(u) = -\sum_{i=0}^{I} \sum_{n=0}^{\infty} \frac{d^n}{du^n} \left[G_{n+1}^{s}(u)\phi_{k}^{s}(u) \right] , \tag{A.2}$$

where

$$G^n_{\ell,i}(u) \equiv \sum_{\ell'=0}^{\infty} \Sigma^{\ell'}_{s_i}(u) T_{\ell\ell',i}$$

$$\equiv \sum_{\ell'=0}^{\infty} \Sigma^{\ell'}_{s_i}(u) \frac{2\ell' + 1}{2n!} \int_{n-\beta_i}^{u} du' \int_{u'}^{u'+\beta_i} du'' \, P_\ell\left(\mu_C(u' - u'')\right)$$

$$\times P_\ell\left(\mu_C(u' - u'')\right)(u' - u)^n \frac{2e^{u'-u''}}{1 - \alpha_i} . \tag{A.3}$$

Differentiating Eq. (A.2)

$$\frac{dq_\ell(u)}{du} = - \sum_i \sum_{n=0}^{\infty} \frac{d^{n+1}}{du^{n+1}} \left(G^{n+1}_{\ell,i}(u)\phi_\ell(u)\right) \tag{A.4}$$

and combining Eqs. (A.2) and (A.4) results in

$$q_\ell(u) + \lambda_\ell(u) \frac{dq_\ell(u)}{du} = - \sum_i \left[G^1_{\ell,i}(u)\phi_\ell(u) \left\{ 1 - \frac{d\lambda_\ell(u)}{du} \right\} \right.$$

$$\left. + \sum_{n=2}^{\infty} \left\{ \frac{d^n G^{n+1}_{\ell,i}(u)\phi_\ell(u)}{du^n} + \lambda_\ell(u) \frac{d^n G^n_{\ell,i}(u)\phi_\ell(u)}{du^n} \right\} \right] , \tag{A.5}$$

where λ_ℓ has been chosen to eliminate first derivative terms involving ϕ_ℓ,

$$\lambda_\ell(u) \equiv - \frac{\sum_i G^2_{\ell,i}(u)}{\sum_i G^1_{\ell,i}(u)} . \tag{A.6}$$

$$G^{\beta}_{\alpha,i}(u) \equiv \sum_{k''=0}^{\infty} \tau^{\beta}_{k''i}(u) T_{k''i}$$

$$\equiv \sum_{k''=0}^{\infty} \tau^{\beta}_{k''i}(u) \frac{2k'' + 1}{2n!} \int_{u-\beta^{-1}}^{u} du' \int_{u'}^{u'+\beta^{-1}} du'' \, P^{\beta}\left(u^{\alpha}(u' - u'')\right)$$

$$\times P^{\beta}\left(u^{\alpha}(u'' - u')\right)(u' - u)^{n} \frac{2e^{u'-u''}}{1 - e^{-1}} . \tag{A.3}$$

Differentiating Eq. (A.2)

$$\frac{dq^{\beta}(u)}{du} = - \sum_{i} \sum_{n=0}^{\infty} \frac{d^{n+1}}{du^{n+1}} \left(G^{\beta,i}_{n+1}(u)\phi^{\beta}(u)\right) \tag{A.4}$$

and combining Eqs. (A.2) and (A.4) results in

$$q^{\beta}(u) + \gamma^{\beta}(u) \frac{dq^{\beta}(u)}{du} = - \sum_{i} \left[G^{\beta,i}_{1}(u)\phi^{\beta}(u)\right] \left\{ i - \frac{d\gamma^{\beta}(u)}{du} \right\}$$

$$+ \sum_{n=2}^{\infty} \left[\frac{d^{n}G^{\beta}_{n-1}(u)\phi^{\beta}(u)}{du^{n}} + \gamma^{\beta}(u) \frac{d^{n}G^{\beta}_{n}(u)\phi^{\beta}(u)}{du^{n}} \right] , \tag{A.5}$$

where γ^{β} has been chosen to eliminate first derivative terms involving ϕ^{β},

$$\gamma^{\beta}(u) \equiv - \frac{\sum_{i} G^{\beta,i}_{2}(u)}{\sum_{i} G^{\beta,i}_{1}(u)} . \tag{A.6}$$

The conventional Greuling-Goertzel theory [Eqs. (15) through (17)] is obtained by neglecting the terms in the sum $n \geq 2$ and identifying the parameters defined by Eqs. (16) and (17)

$$\xi_\ell^i(u) \equiv -G_{\ell,i}^1(u)/\Sigma_s(u) , \tag{A.7}$$

$$a_\ell^i(u) \equiv -G_{\ell,i}^2(u)/\Sigma_s(u) . \tag{A.8}$$

The original Greuling-Goertzel theory[1,5] utilized separate equations of the form of Eqs. (15) [or Eqs. (A.5)] for each isotope (with an isotopic ξ, a, q, and Σ_s) in a mixture. Ferziger and Zweifel[9] suggested the use of composite moderating parameters and a total elastic slowing-down density, and Williams[10] explicitly defined composite moderating parameters for the case of isotropic scattering. The above derivation extends Williams'[10] definitions to the case of anisotropic scattering, and reduces to Amster's[4] results for the case of a single isotope.

Amster[14] mode estimates for the accuracy of the Greuling-Goertzel approximation and the sensitivity to various Legendre moments of the angular scattering data based on the THEOREM: As γ (γ = inverse atomic mass) approaches zero, $T_{\ell\ell'}^n$ approaches zero at least as fast as the faster of γ^u and $\gamma^{|\ell-\ell'|}$. Generalizing his analysis to a mixture, it is estimated that λ_ℓ, given by

$$\lambda_\ell(u) \equiv \frac{\sum_i \sum_{\ell'=0}^{\infty} T_{\ell\ell',i}^2 \Sigma_{s_i}^{\ell'}(u)}{\sum_i \sum_{\ell'=0}^{\infty} T_{\ell\ell',i}^1 \Sigma_{s_i}^{\ell'}(u)} ,$$

The conventional Gerling-Goertzel theory [Eqs. (15) through (17)] is obtained by replacing the terms in the sum $n \geq 2$ and identifying the parameters defined by Eqs. (16) and (17)

$$t_{\ell}^{i}(u) \equiv -G_{\ell,i}^{1}(u)/\xi_{s}(u) \; , \tag{A.7}$$

$$s_{\ell}^{i}(u) \equiv -G_{\ell,i}^{2}(u)/\xi_{s}(u) \; . \tag{A.8}$$

The original Gerling-Goertzel theory[1,5] utilized separate equations of the form of Eqs. (15) for each isotope (with an isotopic ξ, s, q, and λ) in a mixture. Berdiyev and Zweifel[8] suggested the use of composite moderating parameters and a total elastic slowing-down density, and Williams[10] explicitly defined composite moderating parameters for the case of isotropic scattering. The above derivation extends Williams'[10] definitions to the case of anisotropic scattering, and reduces to Amster's[4] results for the case of a single isotope.

Amster[1,4] made estimates for the accuracy of the Gerling-Goertzel approximation and the sensitivity to various Legendre moments of the angular scattering data based on the THEOREM: As γ (γ = inverse atomic mass) approaches zero, $T_{\ell\ell}^{n}$ approaches zero at least as fast as the faster of γ^{n} and $\gamma^{|\ell-\ell'|}$. Generalizing the analysis to a mixture, it is estimated that λ_{ℓ}^{i}, given by

$$\lambda_{\ell}^{i}(u) \equiv \frac{\sum_{\ell'=0}^{\infty} T_{\ell\ell'}^{2} \, \xi_{\ell'}^{-1} \, \xi_{\ell'}^{i}(u)}{\sum_{\ell'=0}^{\infty} T_{\ell\ell'}^{1} \, \xi_{\ell'}^{-1} \, \xi_{\ell'}^{i}(u)} \; ,$$

is proportional to γ_i (the inverse mass) for the lightest isotope in the mixture, unless $\Sigma_{s_i}^{\ell'} = 0$ for $\ell' > \ell - 2$. It may be further argued that the lowest order terms neglected in Eq. (A.5).

$$\frac{d^2 G_\ell^3 \phi_\ell}{du^2} = \sum_{\ell'=0}^{\infty} T_{\ell\ell',i}^3 \frac{d^2 \Sigma_{s_i}^{\ell'} \phi_\ell}{du^2} ,$$

and

$$\lambda_\ell \frac{d^2 G_{\ell,i}^2}{du^2} = \lambda_\ell \sum_{\ell'=0}^{\infty} T_{\ell\ell',0}^2 \frac{d^2 \Sigma_{s_i}^{\ell'} \phi_\ell}{du^2} ,$$

are of order γ_i^3. Continuing in this vein, it may be argued that in evaluating the moderating parameters of Eqs. (A.7) and (A.8), using Eq. (A.3) there is no need to retain terms of order greater than γ_i^2, and thus the sums over ℓ' may be terminated at $\ell' = 2$. This line of reasoning is based solely upon the dependence of the $T_{\ell\ell'}^n$ on the atomic mass, and does not account for the relative order of magnitudes of the $\Sigma_{s_i}^{\ell'}$ and the second derivatives of the $\Sigma_{s_i}^{\ell'} \phi_\ell$, and thus should be considered as suggestive rather than definitive.

The improved Greuling-Goertzel theory developed by Stacey[6,7] can be obtained by substituting Eq. (2) into Eq. (1) and making a Taylor's series expansion in the integrand

$$F_\ell(u') \equiv \Sigma(u')\phi_\ell(u') = \sum_{n=0}^{\infty} \frac{(u' - u)^n}{n!} \frac{d^n F_\ell(u)}{du^n} , \tag{A.9}$$

which yields

$$q_\ell(u) = -\sum_i \sum_{n=0}^{\infty} H_{\ell,i}^n(u) \frac{d^n F_\ell(u)}{du^n} , \tag{A.10}$$

is proportional to γ_i (the inverse mass) for the lightest isotope in the mixture, unless $\xi^{s_i}_{\ell'} = 0$ for $\ell' > \ell - 2$. It may be further argued that the lowest order terms neglected in Eq. (A.5),

$$\frac{d^2 G^{\ell}_{\ell}\ \phi_{\ell}}{du^2} = \sum_{\ell'=0}^{\infty} T^{\ell}_{\ell'+1}\ \frac{d^2 \xi^{s_i}_{\ell'} \phi_{\ell'}}{du^2} ,$$

and

$$\gamma_{\ell}\ \frac{d^2 G^{\ell}_{\ell}\ i}{du^2} = \gamma_{\ell} \sum_{\ell'=0}^{\infty} T^{\ell}_{\ell'+0}\ \frac{d^2 \xi^{s_i}_{\ell'} \phi_{\ell'}}{du^2}$$

are of order γ_i^2. Continuing in this vein, it may be argued that in evaluating the moderating parameters of Eqs. (A.7) and (A.8), using Eq. (A.3) there is no need to retain terms of order greater than γ_i^2, and thus the sums over ℓ' may be terminated at $\ell' = 2$. This line of reasoning is based solely upon the dependence of the $T^{m}_{\ell\ell'}$, on the atomic mass, and does not account for the relative order of magnitudes of the $\xi^{s_i}_{\ell'}$ and the second derivatives of the $\xi^{s_i}_{\ell'} \phi_{\ell'}$, and thus should be considered as suggestive rather than definitive.

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which yields

$$\phi_{\ell}(u) = -\sum_{i} \sum_{n=0}^{\infty} H^{n}_{\ell,i}(u)\ \frac{d^n F_{\ell}(u)}{du^n} , \tag{A.10}$$

where,

$$H^n_{\ell,i}(u) \equiv -\frac{1}{n!}\sum_{\ell'=0}^{\infty}\frac{2\ell'+1}{2}\int_{u-\beta_i}^{u} du' \int_{u}^{u'+\beta_i} du'' \, P_\ell\left(\mu_L(u'-u'')\right)$$

$$P_{\ell'}\left(\mu_C(u'-u'')\right)\frac{2e^{u'-u''}}{1-\alpha_i}\frac{\Sigma_{s_i}^{\ell'}(u')}{\Sigma(u')}(u'-u)^n \, . \qquad \text{(A.11)}$$

Note that $H^n_{\ell,i}$ differs from $G^n_{\ell,i}$ only by the presence of the cross section ratio in the integrand of the former.

Differentiating Eq. (A.10) results in

$$\frac{dq_\ell(u)}{du} = -\sum_i\left[\sum_{n=0}^{\infty} H_{\ell,i}(u)\frac{d^{n+1}F_\ell(u)}{du^{n+1}} + \sum_{n=0}^{\infty}\frac{dH^n_{\ell,i}(u)}{du}\cdot\frac{d\,F_\ell(u)}{du^n}\right] . \qquad \text{(A.12)}$$

Combining Eqs. (A.10) and (A.12) yields

$$q_\ell(u) + \hat{\lambda}_\ell(u)\frac{dq_\ell(u)}{du} = -\sum_i\left[\left\{H^0_{\ell,i}(u) + \hat{\lambda}_\ell(u)\frac{dH^0_{\ell,i}(u)}{}\right\}F_\ell(u)\right.$$

$$+ \sum_{n=2}^{\infty}\left\{H^n_{\ell,i}(u)\frac{d^nF_\ell(u)}{du^n} + \hat{\lambda}_\ell(u)H^{n-1}_{\ell,i}(u)\frac{d^nF_\ell(u)}{du^n}\right.$$

$$\left.\left. + \hat{\lambda}_\ell(u)\frac{dH^n_\ell(u)}{du}\frac{d^nF_\ell(u)}{du^n}\right\}\right] , \qquad \text{(A.13)}$$

where

where

$$H^{\varepsilon}_{n+1}(u) = -\frac{1}{n!}\sum_{k'=0}^{\infty}\frac{2k'+1}{2}\int_{u-\varepsilon^{1}}^{u} du'\int_{u'+\varepsilon^{1}}^{u} du'' P_{k'}\left(u'-u''\right)$$

$$P_{k'}\left[u_0(u'-u'')\right]\frac{2\varepsilon^{n}}{1-\alpha^{1}}\frac{\varepsilon^{k'}(u'')}{\varepsilon(u')}(u'-u)^{n} \qquad \text{(A.11)}$$

Note that H^{ε}_{n} differs from G^{ε}_{n} only by the presence of the cross section ratio in the integrand of the former.

Differentiating Eq. (A.10) results in

$$\frac{dg^{\varepsilon}(u)}{du} = -\sum_{i}\left[\sum_{n=0}^{\infty} H^{\varepsilon}_{\varepsilon_i}(u)\frac{d\,F^{\varepsilon}_{n+1}(u)}{du^{n+1}} + \sum_{n=0}^{\infty}\frac{dH^{\varepsilon}_{\varepsilon_i}(u)}{du}\cdot\frac{d\,F^{\varepsilon}_{n}(u)}{du^{n}}\right]. \qquad \text{(A.12)}$$

Combining Eqs. (A.10) and (A.12) yields

$$g^{\varepsilon}(u) + \lambda^{\varepsilon}(u)\frac{dg^{\varepsilon}(u)}{du} = -\sum_{i}\left[\left\{H^{\varepsilon}_{0\,\varepsilon_i}(u) + \lambda^{\varepsilon}(u)\frac{dH^{\varepsilon}_{0\,\varepsilon_i}(u)}{du}\right\}F^{\varepsilon}(u)\right.$$

$$+\sum_{n=2}^{\infty}\left\{H^{\varepsilon}_{n\,\varepsilon_i}(u)\frac{d^{n}F^{\varepsilon}(u)}{du^{n}} + \lambda^{\varepsilon}(u)H^{\varepsilon}_{n-1\,\varepsilon_i}(u)\frac{d^{n}F^{\varepsilon}(u)}{du^{n}}\right.$$

$$\left.\left. + \lambda^{\varepsilon}(u)\frac{dH^{\varepsilon}_{n}(u)}{du}\frac{d^{n}F^{\varepsilon}(u)}{du^{n}}\right\}\right] \qquad \text{(A.13)}$$

where

$$\hat{\lambda}_\ell(u) \equiv - \frac{\sum_i H^1_{\ell,i}(u)}{\sum_i \left[H^0_{\ell,i}(u) + \frac{dH^1_{\ell,i}(u)}{du} \right]} \tag{A.14}$$

has been chosen to eliminate first derivatives of F_ℓ. The theory of Eqs. (4) through (9) is obtained by neglecting the sum $n > 2$ in Eq. (A.13) and identifying the parameters defined in Eqs. (6) through (9)

$$\bar{\xi}^i_\ell(u) \equiv -H^0_{\ell,i}(u) , \tag{A.15}$$

$$\bar{a}^i_\ell(u) \equiv -H^1_{\ell,i}(u) , \tag{A.16}$$

$$c^i_\ell(u) \equiv - \frac{dH^0_{\ell,i}(u)}{du} , \tag{A.17}$$

$$e^i_\ell(u) \equiv H^0_{\ell,i}(u) + \frac{dH^1_{\ell,i}(u)}{du} . \tag{A.18}$$

Because of the functional similarity of the improved and conventional Greuling-Goertzel theories, the estimates made above for the accuracy and sensitivity to Legendre moments of the scattering data may be taken as rough estimates for the improved theory, also.

$$\bar{f}_{\ell}(u) = -\frac{\frac{\ell}{i} H^{1}_{\ell,i}(u)}{\frac{\ell}{i}\left[H^{0}_{\ell,i}(u) + \frac{dH^{1}_{\ell,i}(u)}{du}\right]} \tag{A.14}$$

has been chosen to eliminate first derivatives of f_{ℓ}. The theory of Eqs. (4) through (9) is obtained by neglecting the sum $n > 2$ in Eq. (A.12) and identifying the parameters defined in Eqs. (8) through (9).

$$\bar{c}^{i}_{\ell}(u) = -H^{0}_{\ell,i}(u) , \tag{A.15}$$

$$\bar{d}^{i}_{\ell}(u) = -H^{1}_{\ell,i}(u) , \tag{A.16}$$

$$\bar{c}^{i}_{\ell}(u) = -\frac{dH^{0}_{\ell,i}(u)}{du} , \tag{A.17}$$

$$\bar{e}^{i}_{\ell}(u) = H^{0}_{\ell,i}(u) + \frac{dH^{1}_{\ell,i}(u)}{du} . \tag{A.18}$$

Because of the functional similarity of the improved and conventional Gruning-Oberteil theories, the estimates made above for the accuracy and sensitivity to Legendre moments of the scattering data may be taken as rough estimates for the improved theory, also.

FOOTNOTES

[a]For a sufficiently large L, which was inversely related to the atomic mass, the recursion relations of Ref. 7 failed because of truncation error. The value of L for which truncation error became a problem was much larger than the value of L at which the value of the moderating parameters became asymptotic, particularly for the light and intermediate mass isotopes. $\hat{\xi}_0$ was found to be relatively insensitive to terms for $\ell > 1$, and $\hat{\gamma}_0$ was found to be relatively insensitive to terms for $\ell > 1$, in agreement with the qualitative discussion given in the previous section.

[b]Based upon previous experience with wide scattering resonances,[6] the term

$$\left[1 + \frac{d}{du}\left(\frac{a_0}{\xi_0}\right)\right]$$

in Eq. (23b) was replaced by unity. This is in accord with common usage of the Greuling-Goertzel approximation.

[c]In addition to absolute attenuation, these curves also give relative attenuation; i.e. by renormalizing the results at any energy E*, the attenuation of neutrons slowing down below that energy could be obtained. Thus, the slopes of the curves over a given energy range are more relevant than the absolute attenuation.

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TABLE I

Composition of Sample Mixtures

Isotope	Concentration (atm/cc × 10^{24})		σ_p(b)	σ_{ne}(b)
	No. 1 (core)	No. 2 (blanket)		
^{16}O	0.016	0.021	2.00	0.3
^{23}Na	0.010	0.007	3.00	0.5
^{56}Fe	0.010	0.015	4.00	1.0
^{238}U	0.010	0.011	10.60	1.8
^{239}Pu	0.001	0.00023	10.30	a

[a] $\sigma_{ne}^{239}(b) = 2.5 + 0.45\sqrt{10/E(MeV)}$.

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TABLE II

Scattering Resonance Parameters

^{56}Fe			^{16}O		
E (MeV)	g	Γ_n (MeV)	E (MeV)	g	Γ_n (MeV)
0.028	1.	1.60-3	0.442	2.	4.60-2
0.074	1.	4.25-4	1.000	2.	1.00-1
0.084	1.	1.00-3	1.312	2.	4.20-2
0.124	1.	1.50-4	1.910	1.	3.00-2
0.130	1.	5.00-4	2.370	1.	1.20-1
0.141	1.	2.30-3	3.600	2.	8.00-1
0.168	1.	6.90-4	3.770	3.	2.50-2
0.188	1.	3.16-3	4.050	1.	9.50-2
0.220	1.	1.30-3	4.200	2.	8.00-2
0.244	1.	3.00-4	4.320	2.	6.00-2
0.273	1.	3.50-3	4.450	1.	2.80-1
0.315	1.	5.50-3			
0.360	1.	9.30-3			
0.382	1.	1.00-2			
0.406	1.	2.50-3			
0.438	1.	1.50-3			
0.470	1.	1.50-3			
0.500	1.	2.50-3			
0.614	1.	2.00-3			
0.645	1.	3.00-3			

TABLE II

Scattering Resonance Parameters

^{59}Fe			^{60}Co		
E (MeV)	S	Γ_n (MeV)	E (MeV)	S	Γ_n (MeV)
0.028	1.	1.86-3	0.442	2.	4.80-2
0.074	1.	4.25-4	1.000	2.	1.00-1
0.084	1.	1.00-3	1.317	2.	4.20-2
0.124	1.	1.50-4	1.720	1.	3.00-2
0.130	1.	5.00-4	2.370	1.	1.20-1
0.141	1.	2.40-3	3.600	2.	8.00-1
0.158	1.	6.90-4	3.770	3.	2.80-2
0.189	1.	3.10-3	4.080	1.	9.50-2
0.220	1.	1.40-3	4.200	2.	8.00-2
0.244	1.	8.00-4	4.720	2.	8.00-2
0.273	1.	3.50-3	4.450	1.	2.80-1
0.315	1.	5.40-3			
0.360	1.	6.20-2			
0.382	1.	1.00-2			
0.406	1.	2.50-3			
0.433	1.	1.40-3			
0.470	1.	1.20-2			
0.500	1.	2.50-3			
0.614	1.	2.00-3			
0.845	1.	3.00-3			

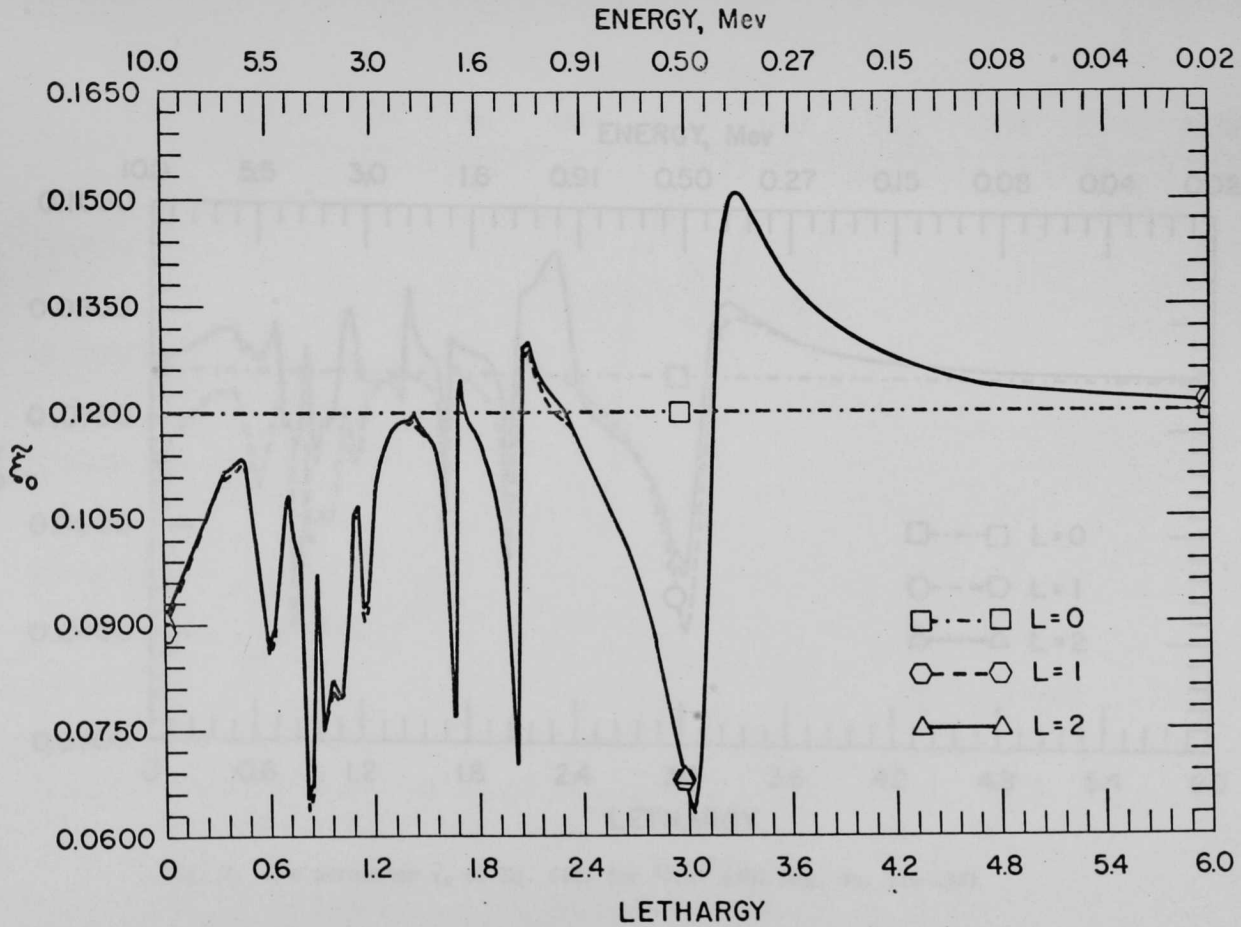


Fig. 1. The parameter $\tilde{\xi}_0$ of Eq. (30) for ^{16}O. (ANL Neg. No. 116-186)

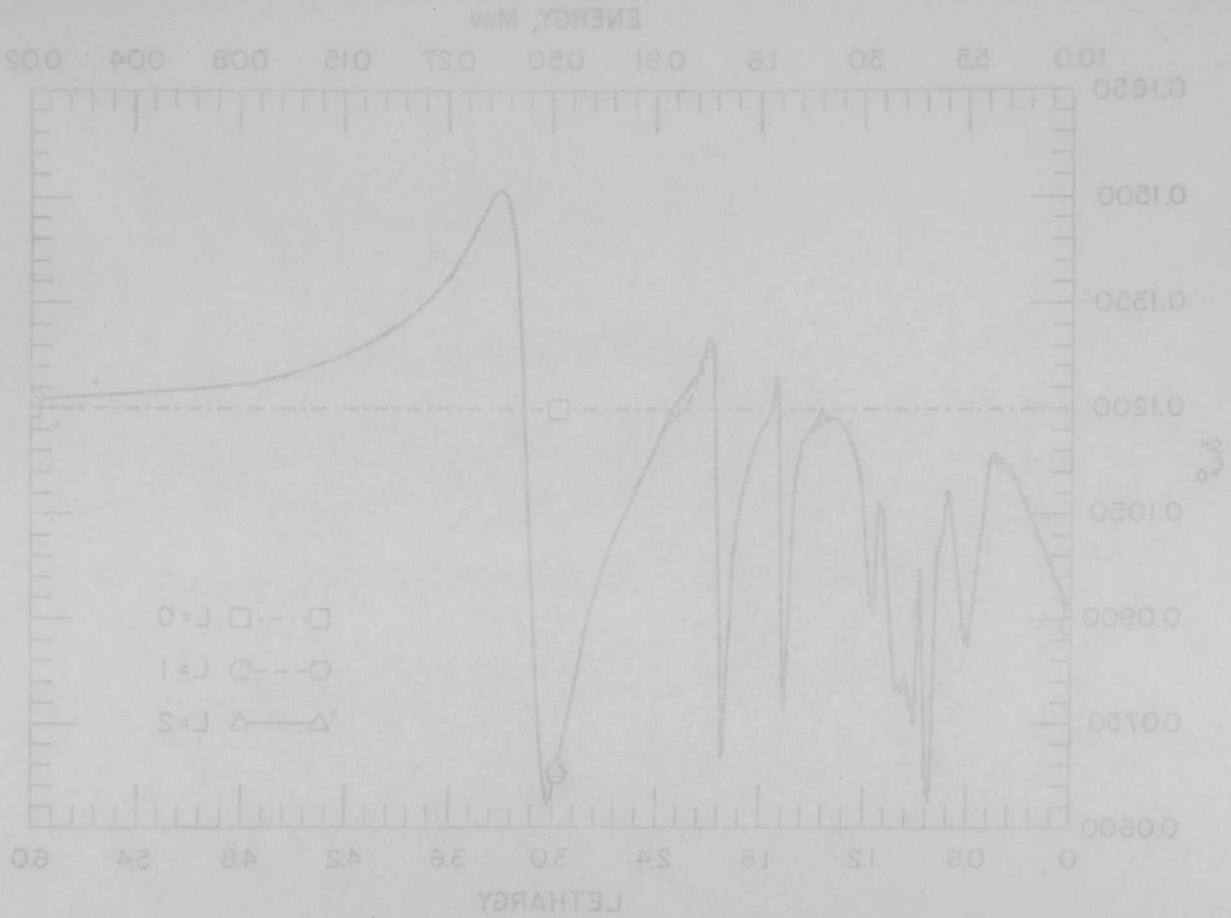


Fig. 1. The parameters δ_0 of Eq. (20) for ^{16}O. (MNL Neg. No. 116-123)

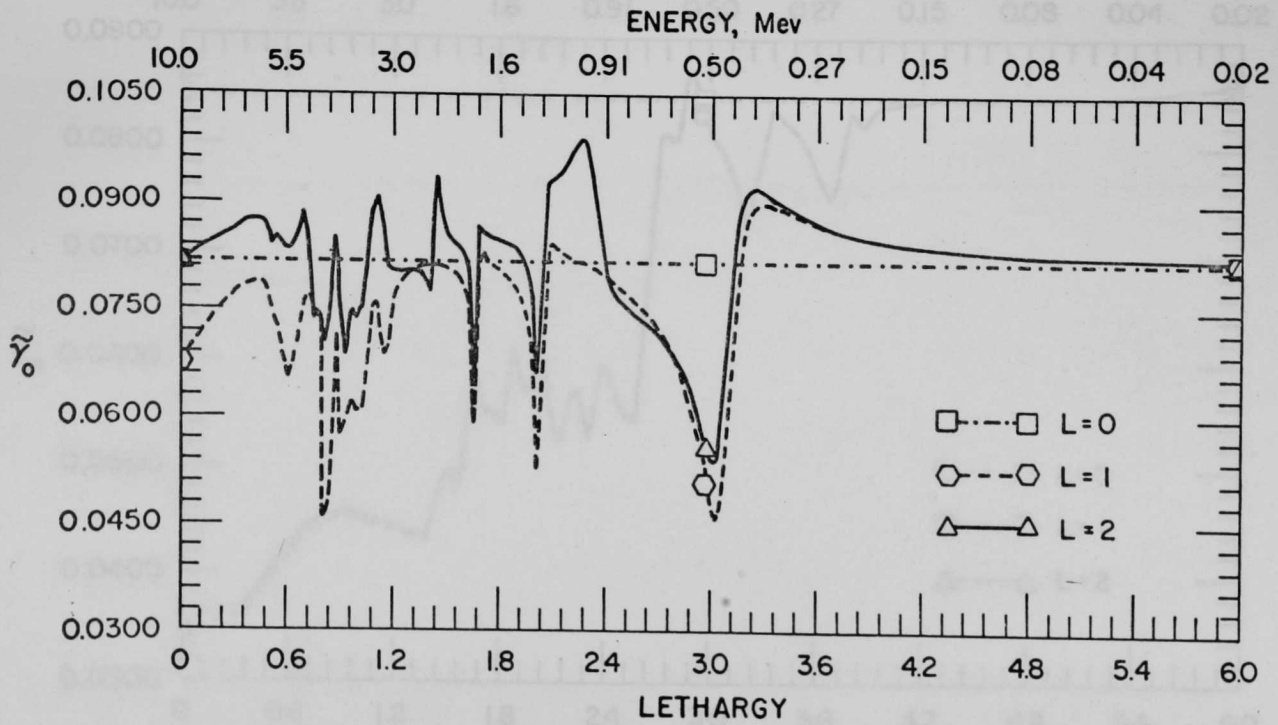


Fig. 2. The parameter $\tilde{\gamma}_0$ of Eq. (31) for ^{16}O. (ANL Neg. No. 116-188)

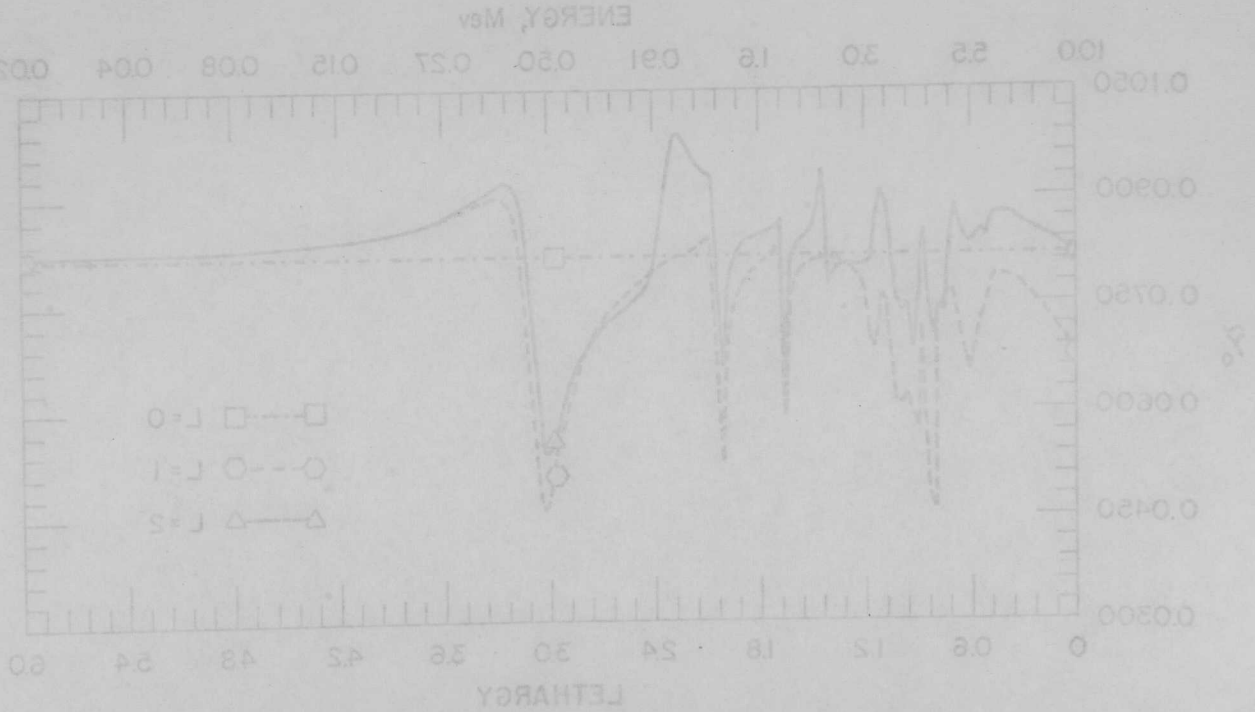


Fig. 2. The parameter η_0 of Eq. (11) for ^{10}B. (Nucl. Sci. Eng., No. 116-1989)

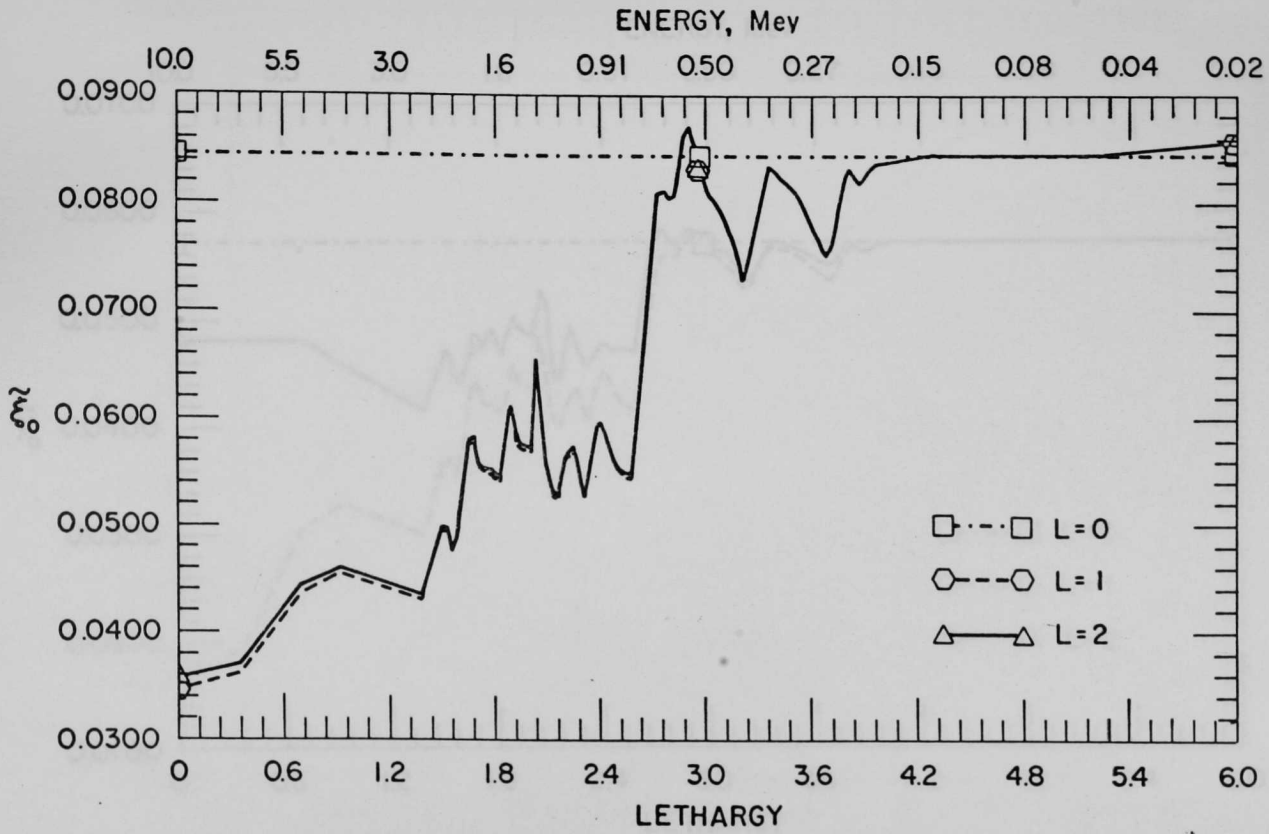


Fig. 3. The parameter $\tilde{\xi}_0$ of Eq. (30) for ^{23}Na. (ANL Neg. No. 116-195)

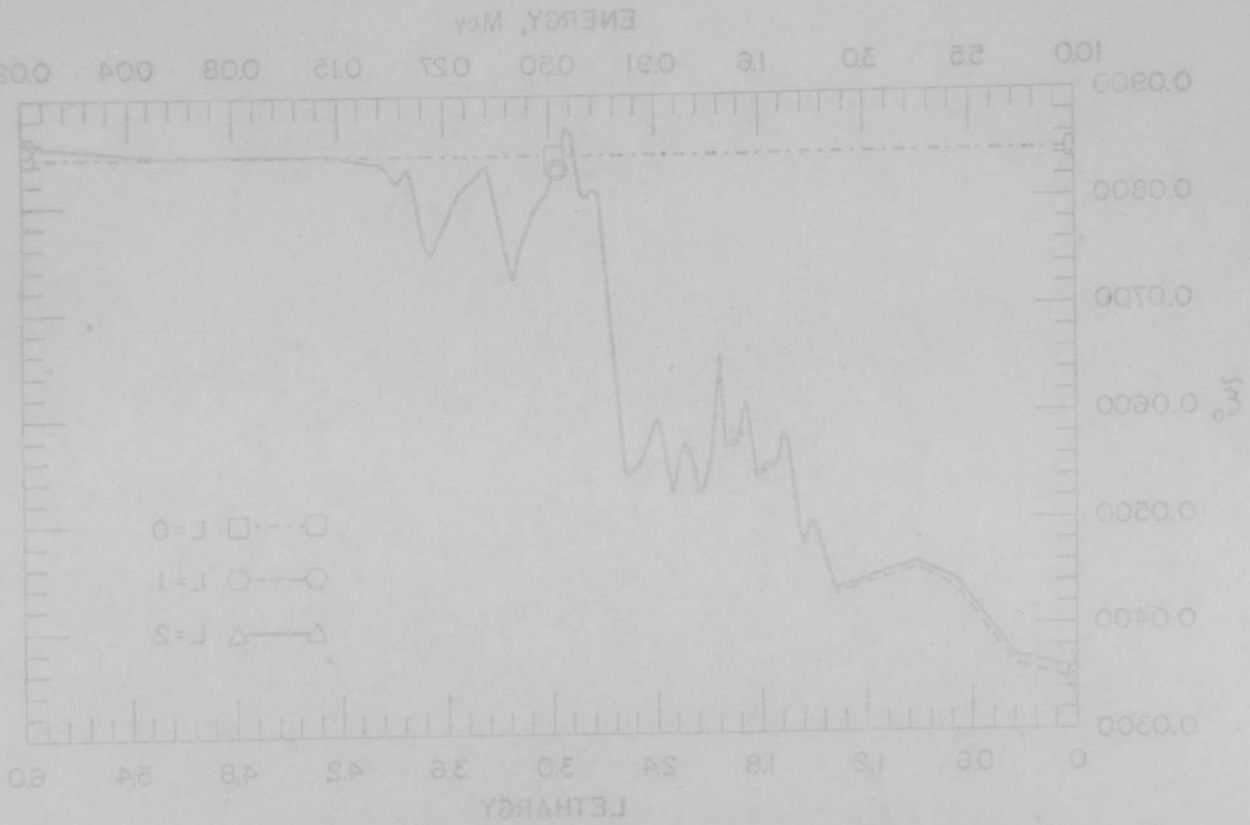


Fig. 4. The parameter ξ_0 of ^{12}C, (2D) for ^{235}U. (Nil. Neg. No. 110-185)

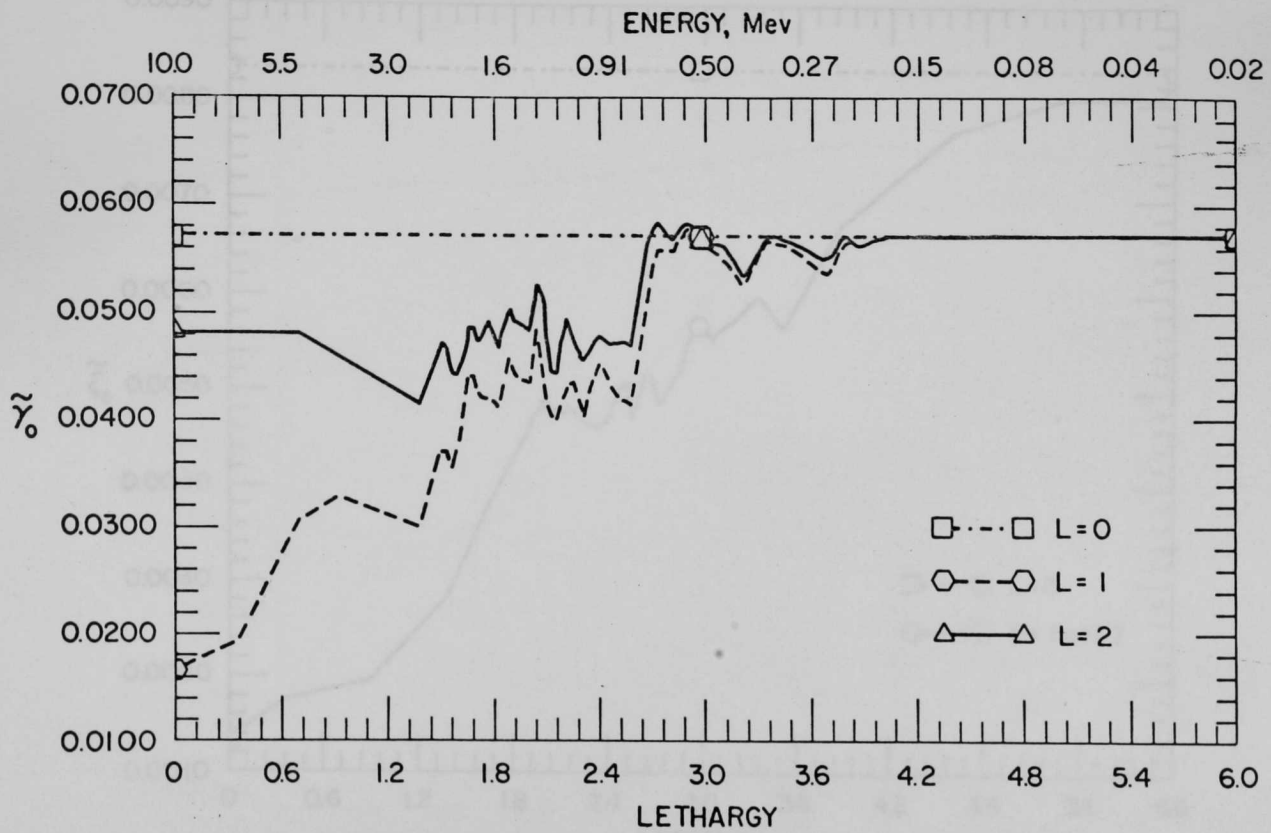


Fig. 4. The parameter $\tilde{\gamma}_0$ of Eq. (31) for ^{23}Na. (ANL Neg. No. 116-192)

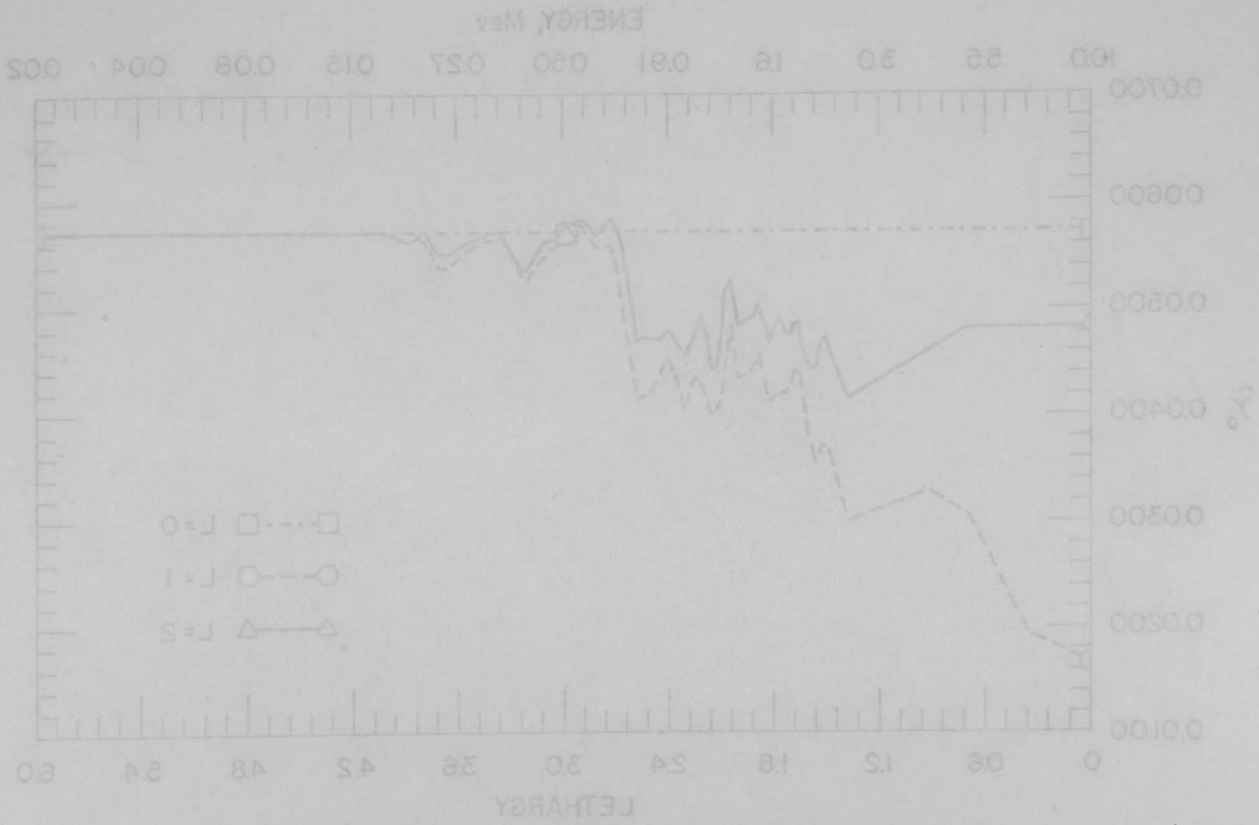


Fig. 9. The parameter $\bar{\eta}_0$ of ^{1}n. (21) for ^{27}Si. (NIL Rep. No. 116-102)

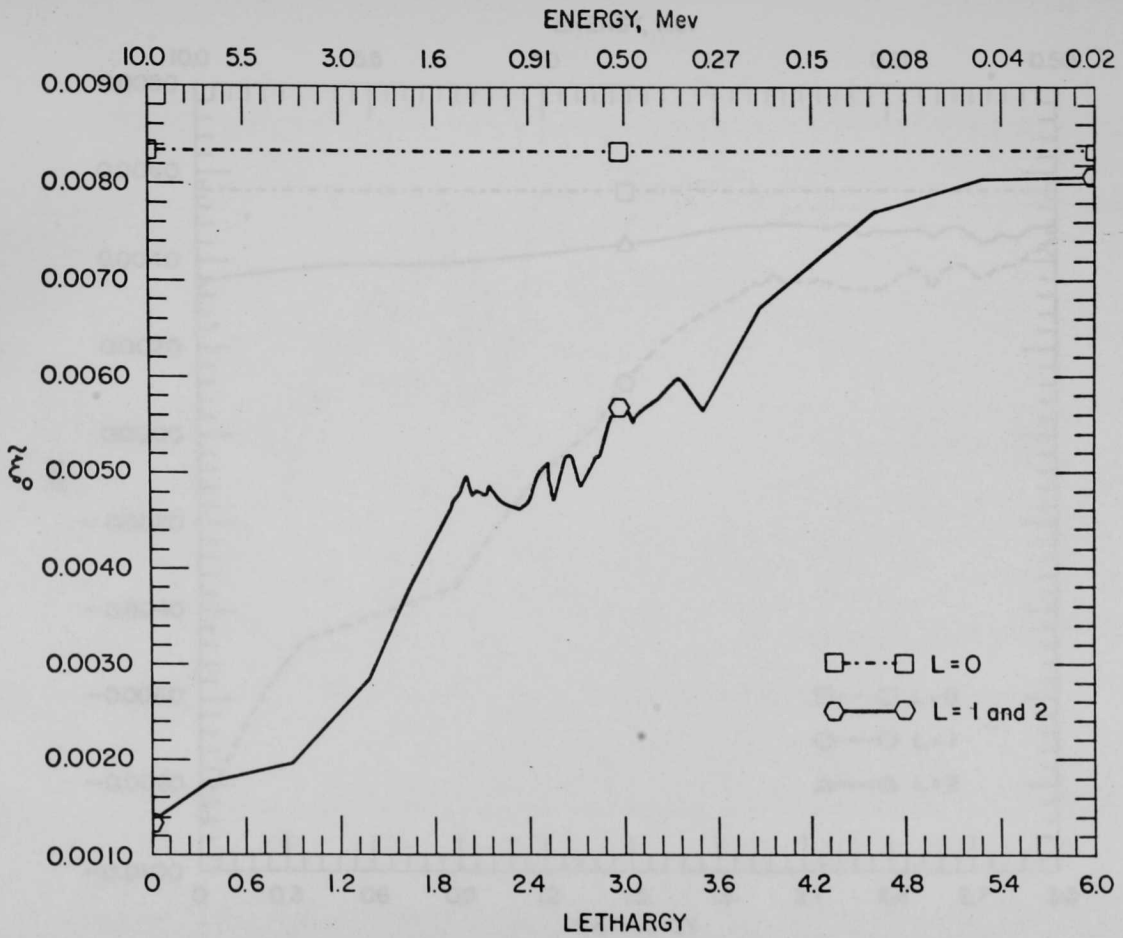
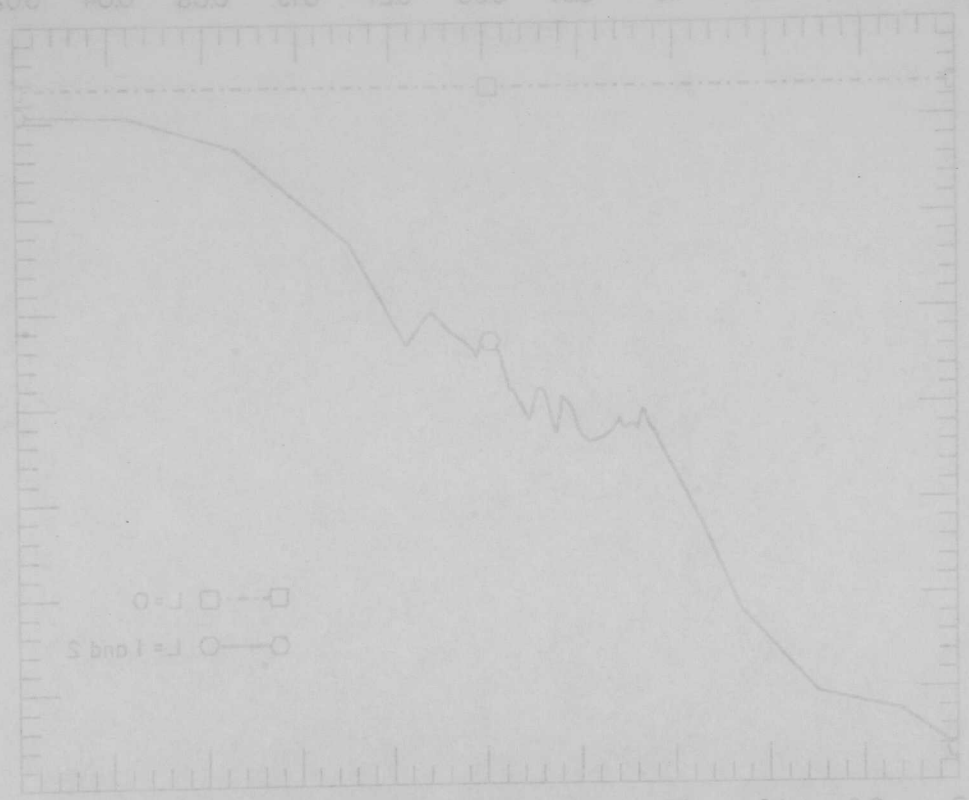


Fig. 5. The parameter $\tilde{\xi}_0$ of Eq. (30) for ^{239}Pu. (ANL Neg. No. 116-187)


L = 0
L = 1 and 2
0.0010
0.0020
0.0030
0.0040
0.0050
0.0060
0.0070
0.0080
0.0090
0
0.6
1.2
1.8
2.4
3.0
3.6
4.2
4.8
5.4
6.0

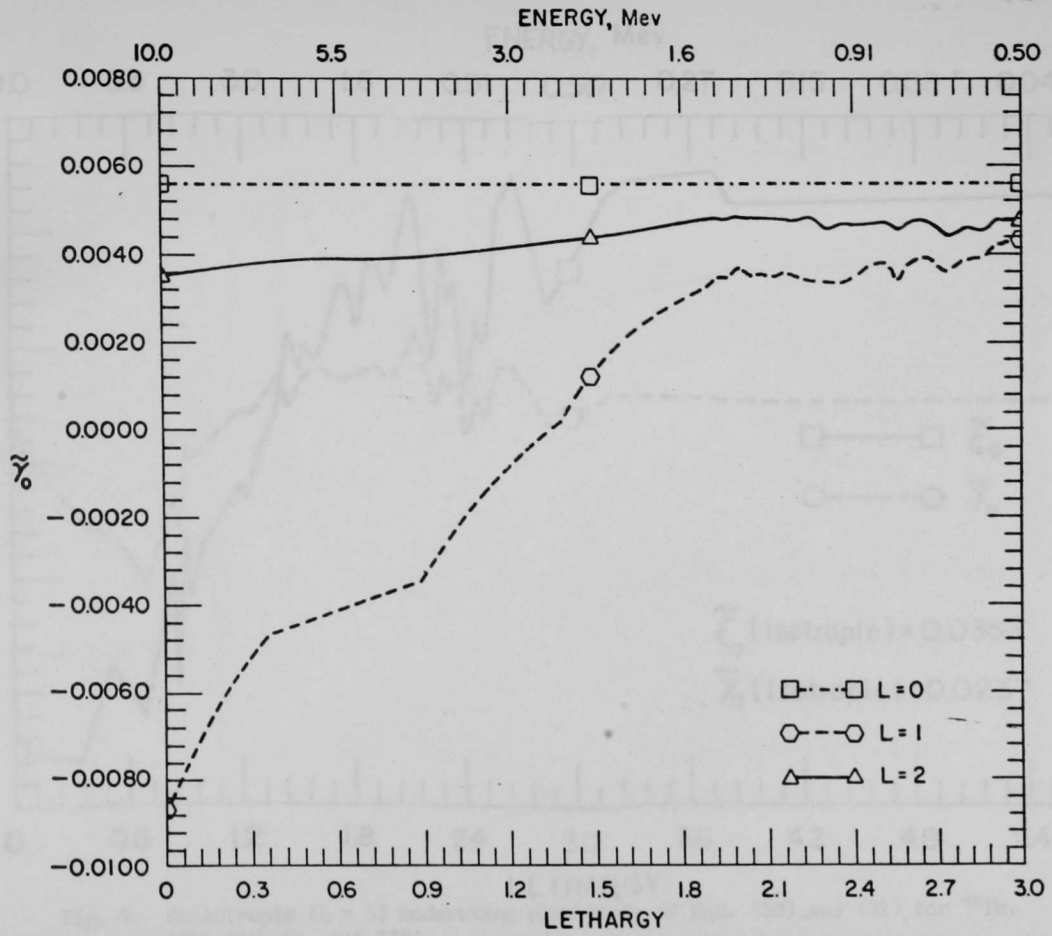


Fig. 6. The parameter $\tilde{\gamma}_0$ of Eq. (31) for ^{239}Pu. (ANL Neg. No. 116-184)

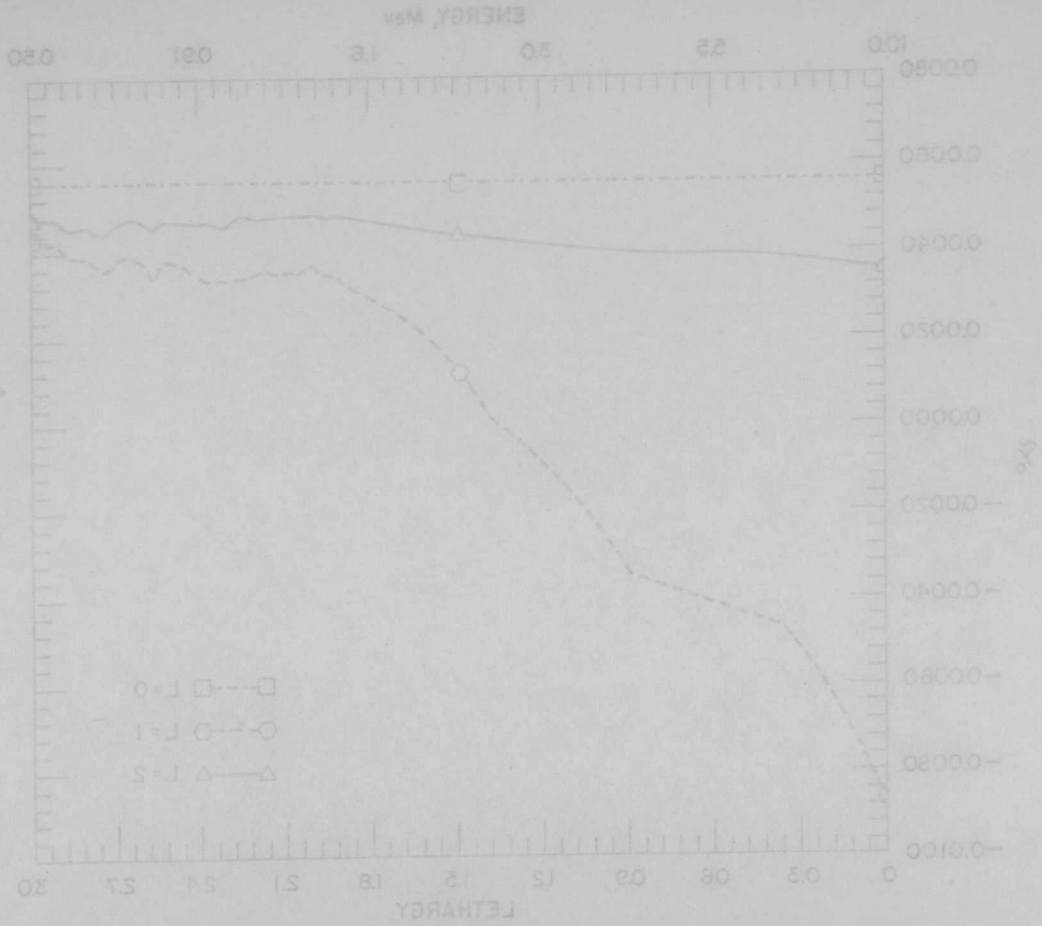


Fig. 6. The parameter γ_0 of Σ_g, (31) for ^{238}U. (AML Neg. No. 115-120)

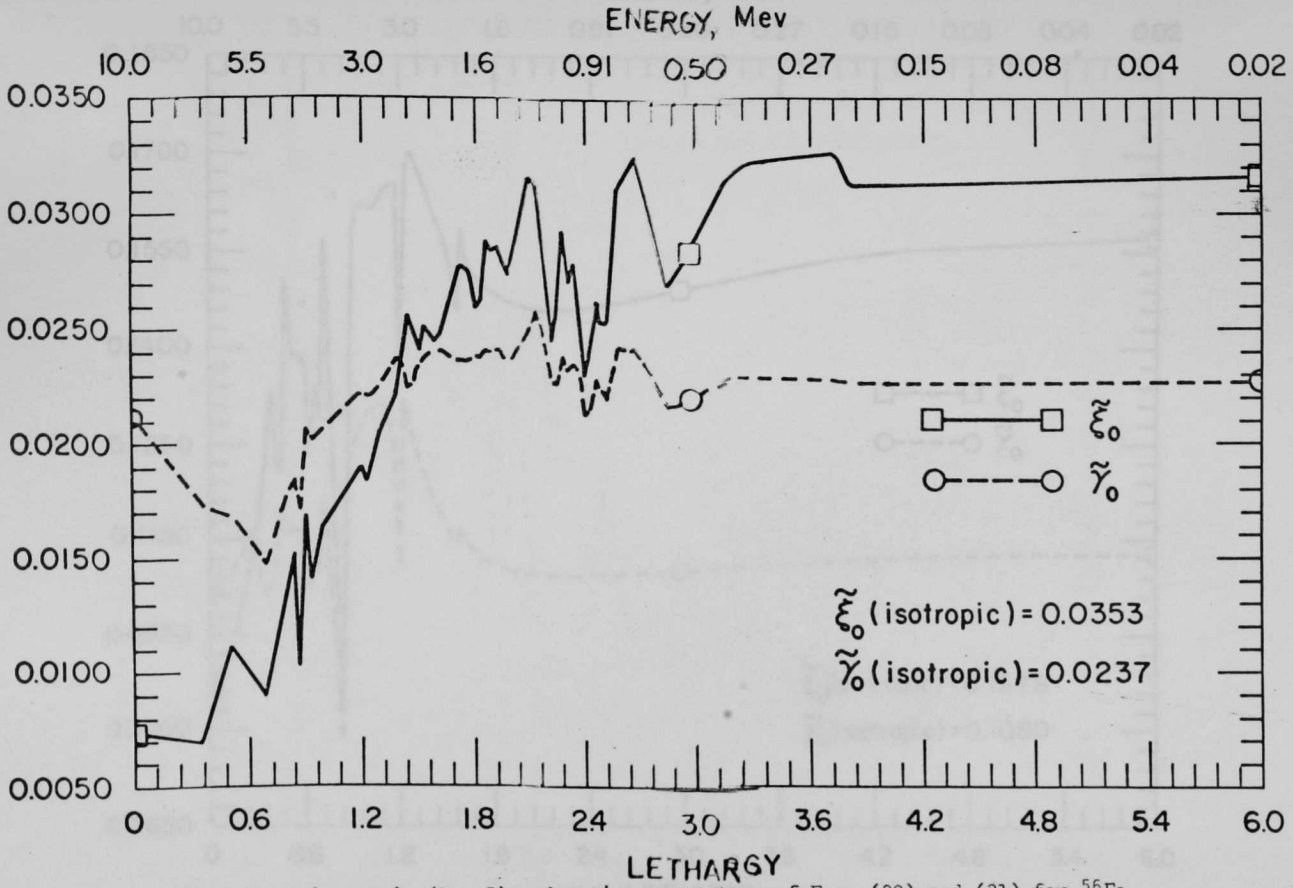


Fig. 7. Anisotropic (L = 5) moderating parameters of Eqs. (30) and (31) for ^{56}Fe. (ANL Neg. No. 116-189)

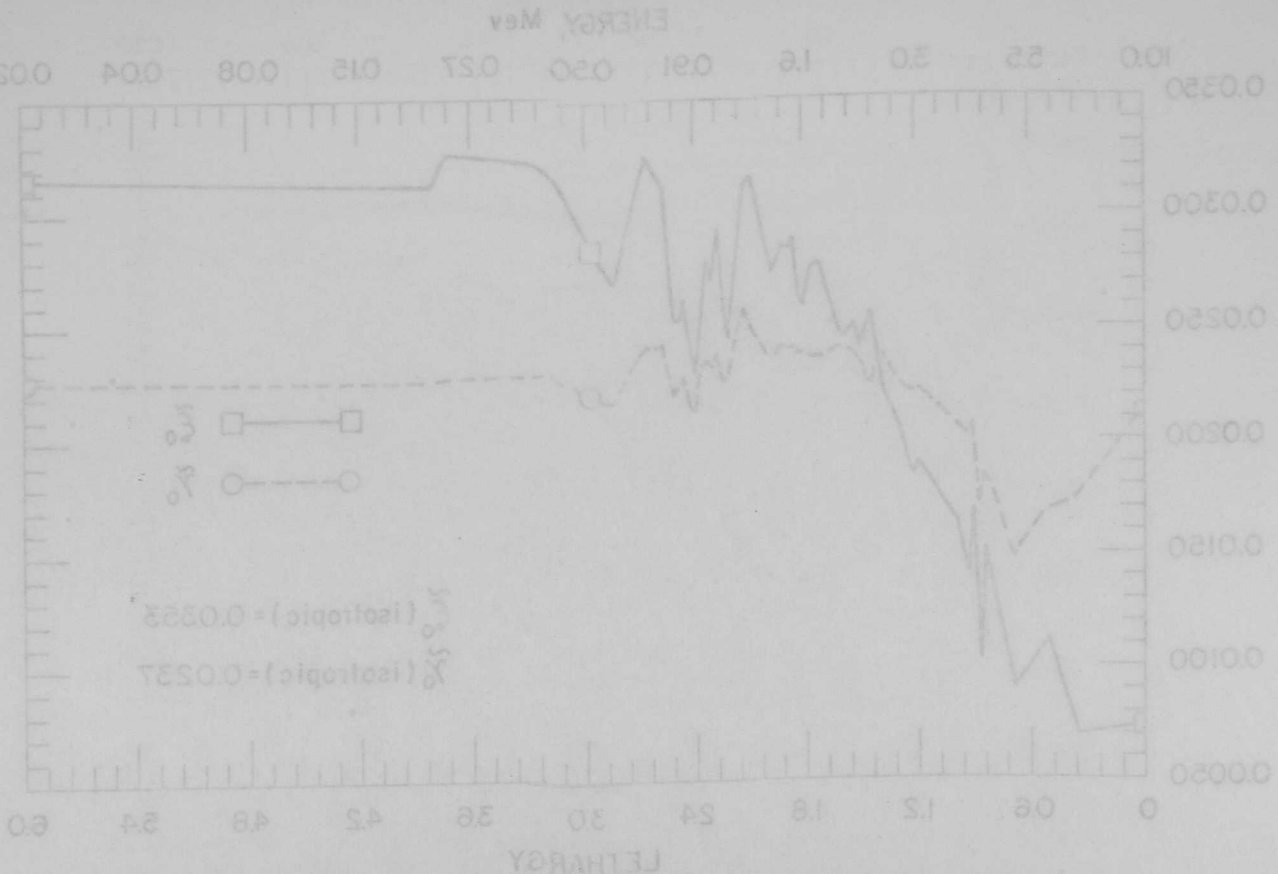


Fig. 7. Anisotropic (L = 5) moderating parameters of ^{12}C, (20) and (21) for ^{56}Fe.
(ML Neg. No. 118-121)

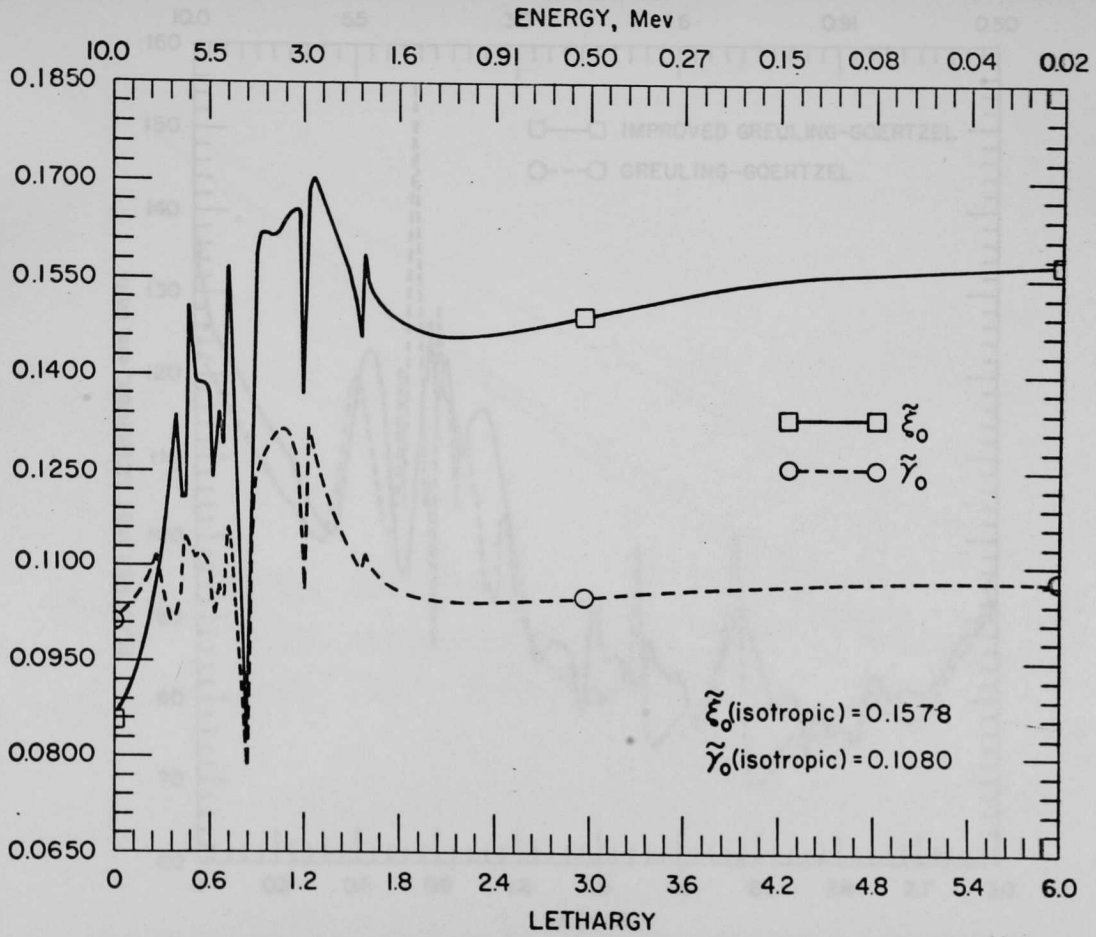
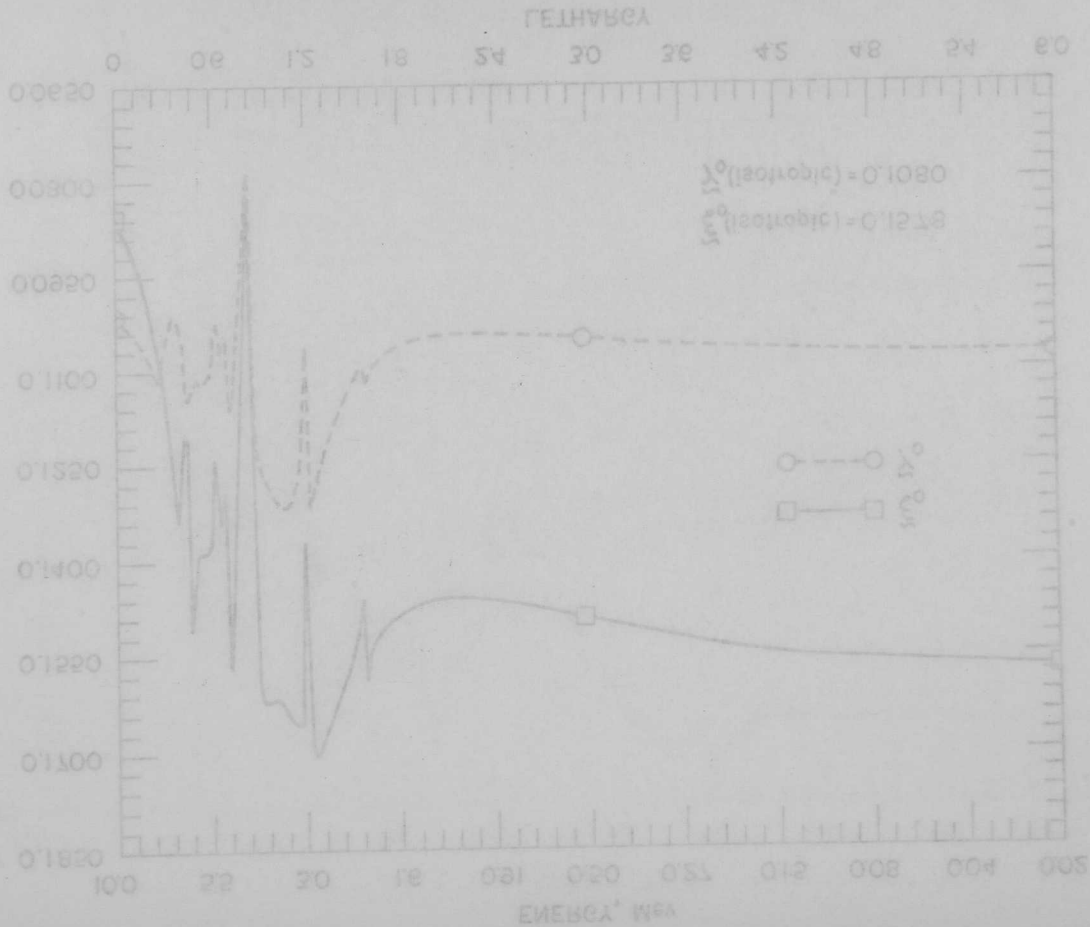


Fig. 8. Anisotropic (L = 5) moderating parameters of Eqs. (30) and (31) for ^{12}C. (ANL Neg. No. 116-190)


0.0
0.4
0.8
1.2
1.6
2.0
2.4
2.8
3.2
3.6
4.0
4.4
4.8
5.2
5.6
6.0
0
0.0020
0.0040
0.0060
0.0080
0.0100
0.0120
0.0140
0.0160
0.0180
0.0200
0.0220
0.0240
0.0260
0.0280
0.0300
0.0320
0.0340
0.0360
0.0380
0.0400
0.0420
0.0440
0.0460
0.0480
0.0500
0.0520
0.0540
0.0560
0.0580
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0.0640
0.0660
0.0680
0.0700
0.0720
0.0740
0.0760
0.0780

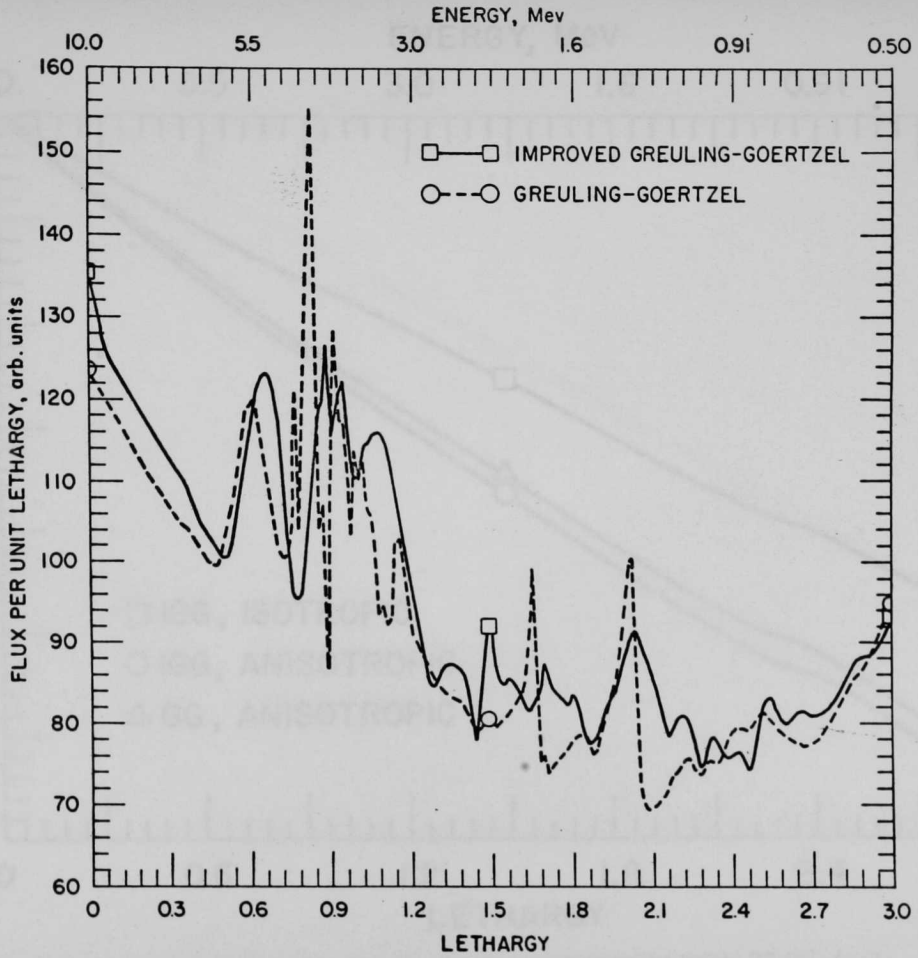
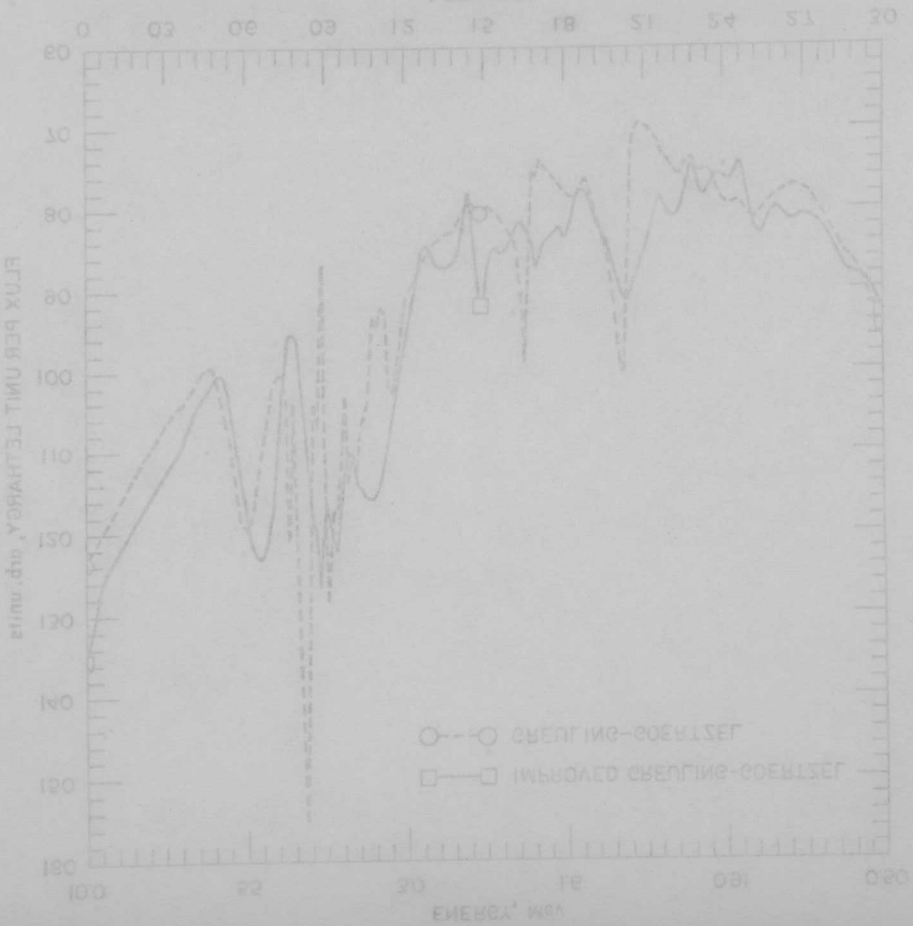


Fig. 9. Effect of anisotropic scattering upon the neutron flux (constant scattering and zero non-elastic sections). (ANL Neg. No. 116-185)


GRENLINE-GOERTZEL
IMPROVED GRENLINE-GOERTZEL
FLUX PER UNIT LETHARGY, arb.units
ENERGY, MeV
LETHARGY

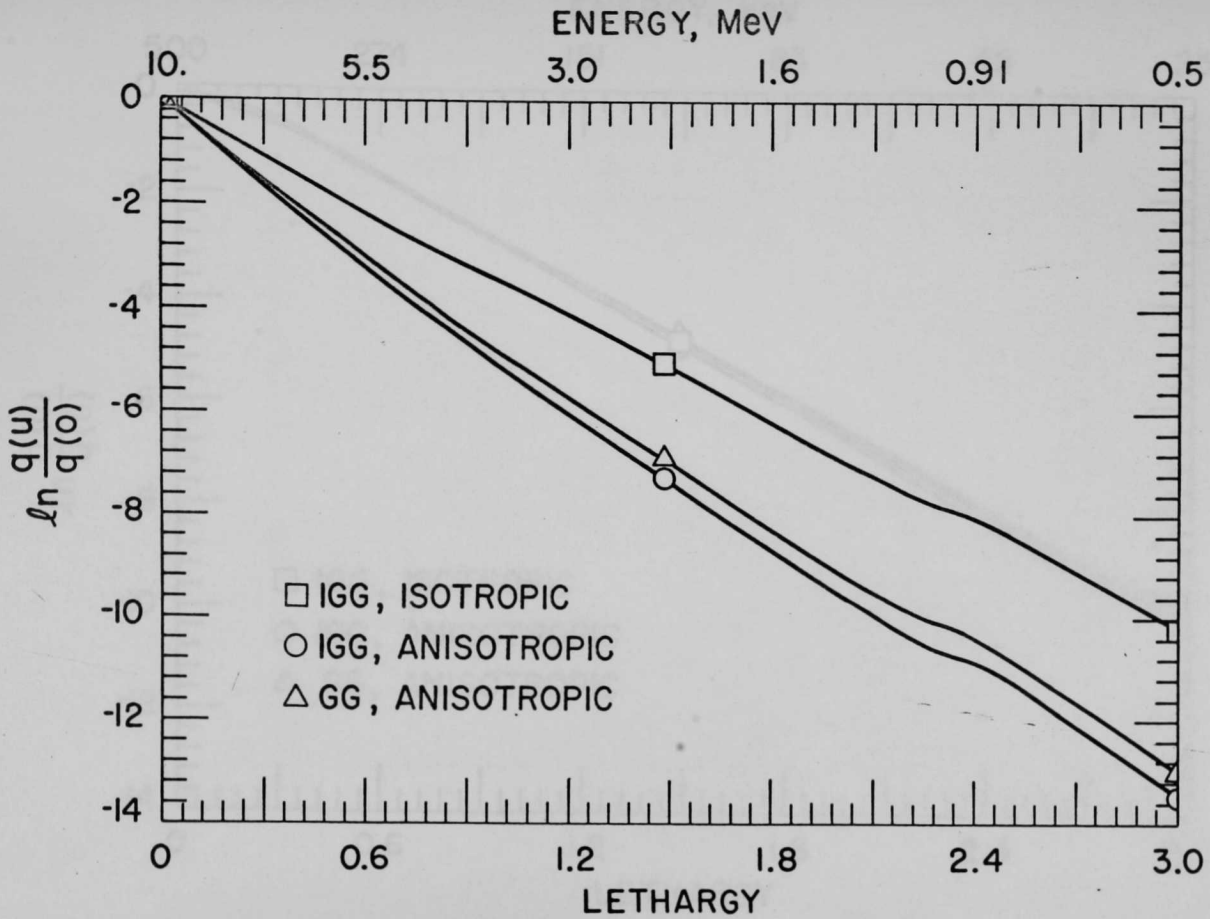


Fig. 10. Attenuation of the elastic slowing-down density below 10 MeV in a typical fast-reactor core. (ANL Neg. No. 116-335)

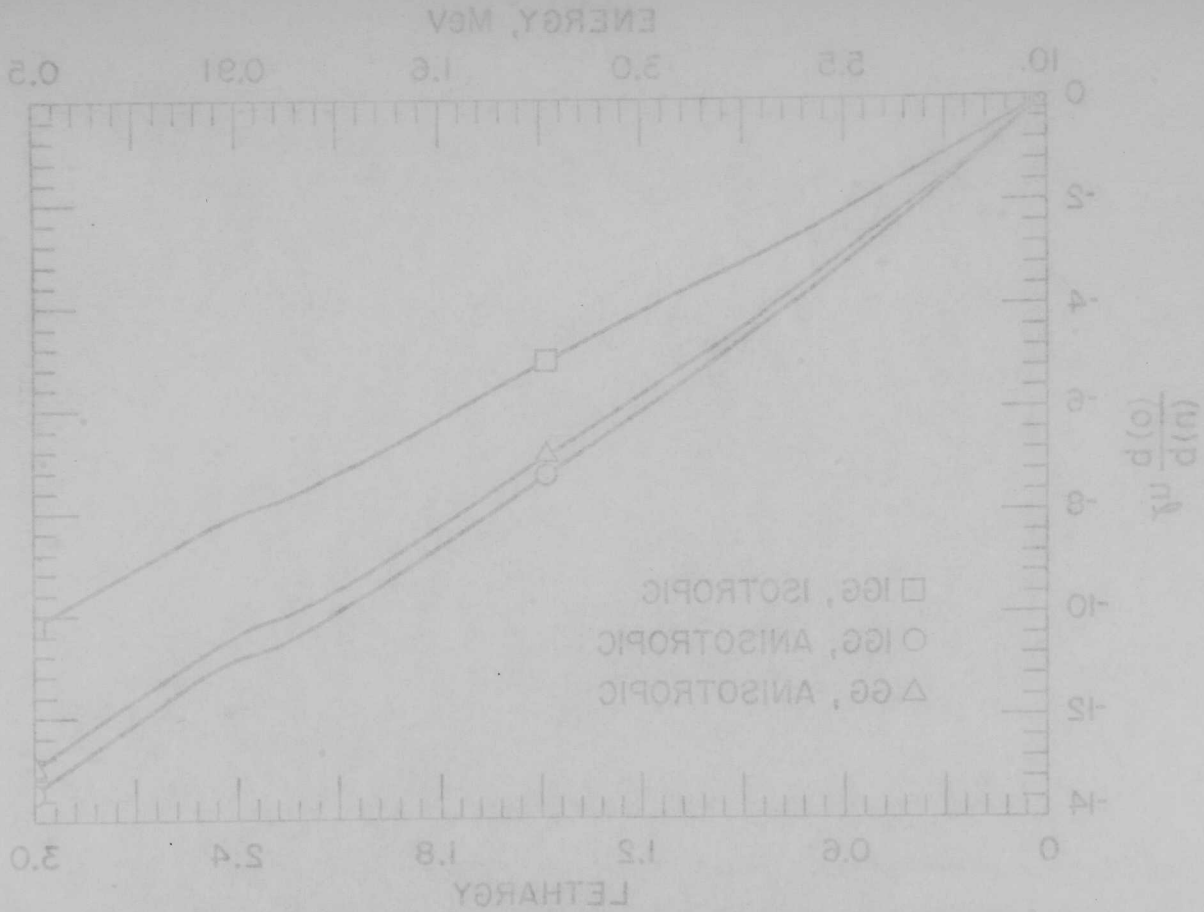


Fig. 10. Attenuation of the elastic fission-spectrum density below 10 MeV in a typical fast-reactor core. (ANL Neg. No. 116-635)

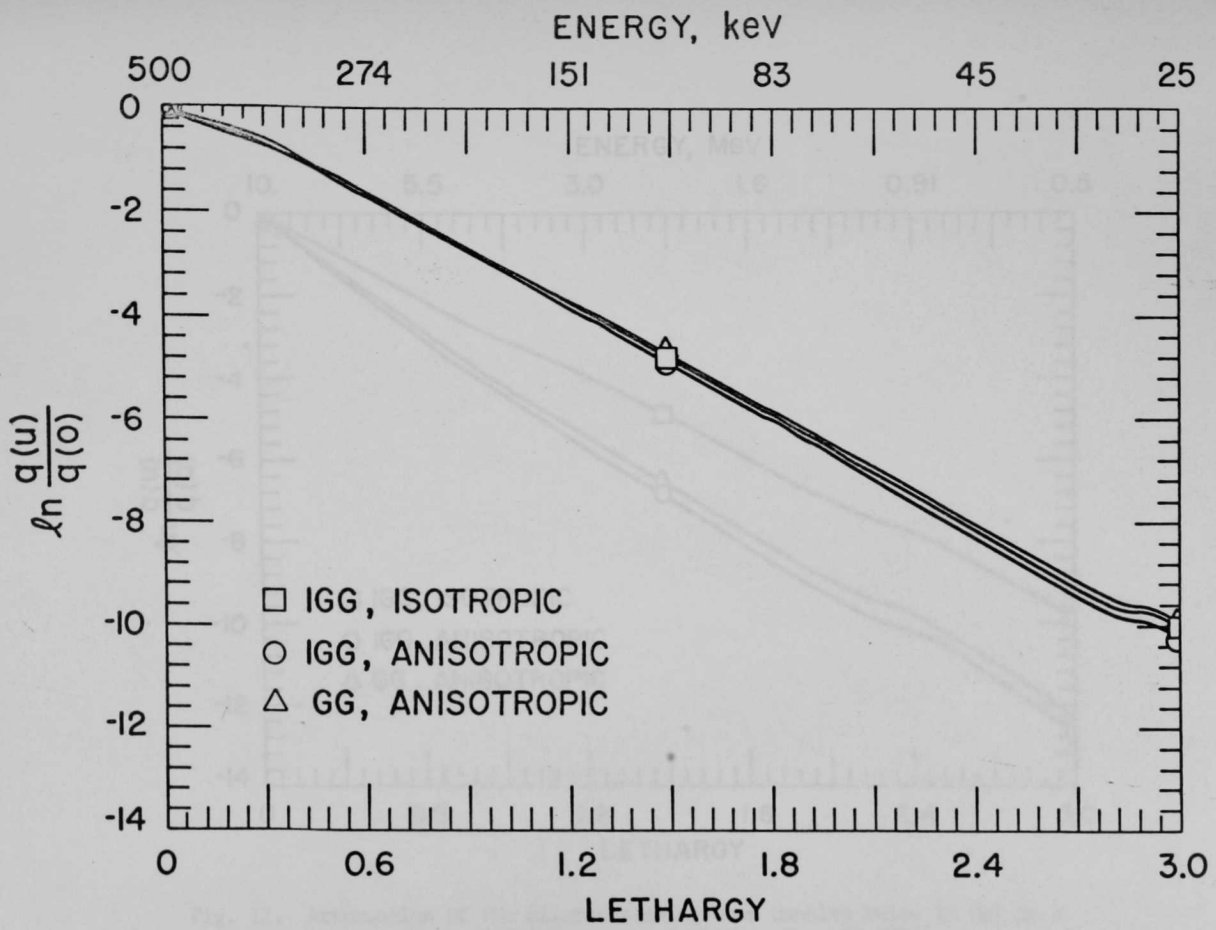


Fig. 11. Attenuation of the elastic slowing-down density below 0.5 MeV in a typical fast-reactor core. (ANL Neg. No. 116-336)

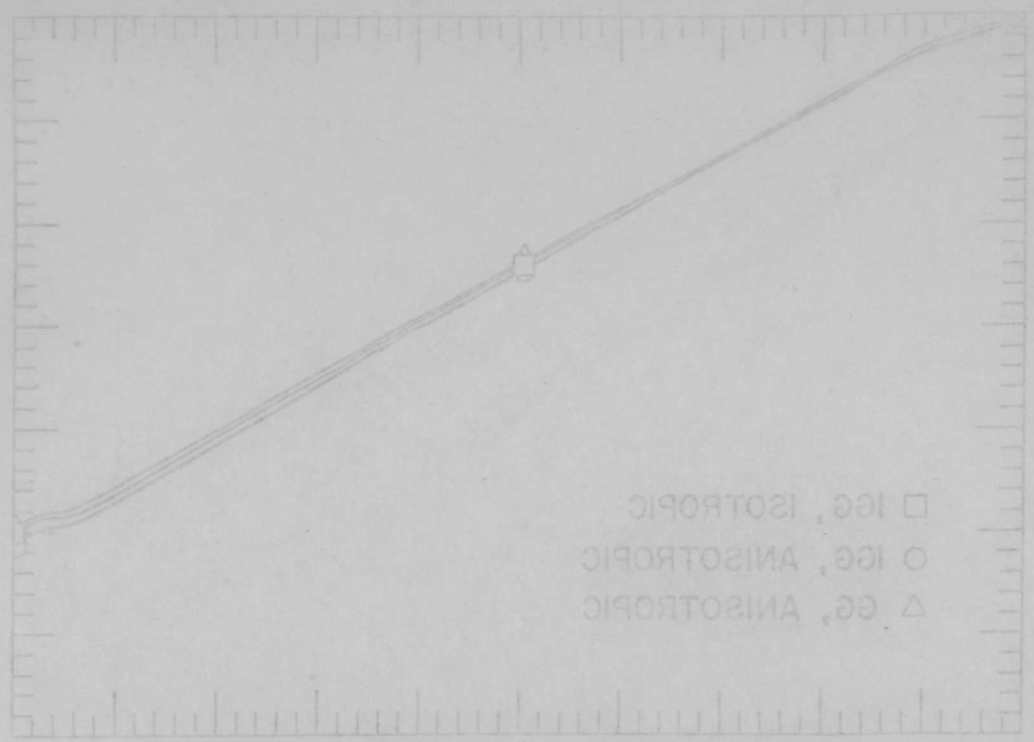


Fig. 11. Attenuation of the elastic slowing-down density below 0.4 Mev in a typical fast-reactor core. (ANL Neg. No. 116-3231)

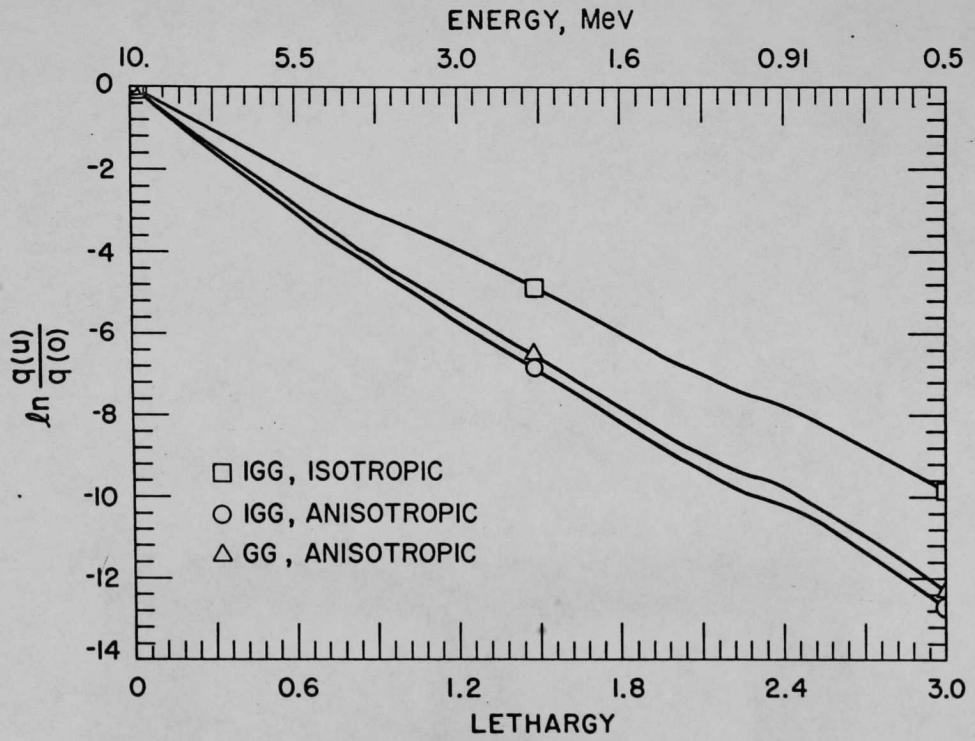


Fig. 12. Attenuation of the elastic slowing-down density below 10 MeV in a typical fast-reactor blanket. (ANL Neg. No. 116-334)

FIG. 11. Attenuation of the energy spectrum (MeV neV, 770-30) of a point-source neutron beam in ice, for anisotropic and isotropic scattering.

B