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‘Granular Elasticity’ and the Loss of Elastic Stability in Granular Materials

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'Granular Elasticity' and the loss of elastic stability in granular materials

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Abstract. A recently proposed hyperelastic model for granular materials, called "granular elasticity", identifies a yield angle as a result of thermodynamic instability. GE gives yield angles that are smaller than those found in real materials; a generalization of the theory is considered here that includes dependence on the third strain invariant. This generalization proves unsuccessful, as it gives smaller, not larger, yield angles. Fully convex hyperelastic models are identified as a point for future investigation.

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INTRODUCTION

In a series of recent papers [1–7], a hyperelastic model for granular statics (called "granular elasticity" or GE) has been proposed, in which the Helmholtz free energy is of the form

$$\mathcal{F} = \mathcal{A}\Delta^a\left(\tfrac{2}{5}\xi\Delta^2 + u_s^2\right) \tag{1}$$

Here A and ξ are material constants, $\Delta \equiv -u_{ii}$, $u_s^2 \equiv u_{ij}^0 u_{ij}^0$, $u_{ij}^0 \equiv u_{ij} - \frac{1}{3}u_{\ell\ell}\delta_{ij}$, and u_{ij} are small *elastic* strains. Δ and u_s^2 are the first and second invariants of the strain tensor,

$$\Delta = -(u_{11}+u_{22}+u_{33}) \tag{2}$$

$$u_s^2 = u_{11}^2+u_{22}^2+u_{33}^2+\tfrac{1}{2}\gamma_{12}^2+\tfrac{1}{2}\gamma_{23}^2+\tfrac{1}{2}\gamma_{13}^2-\tfrac{1}{3}\Delta^2 \tag{3}$$

where γ_{ij} are the "engineering" shear strains, $\gamma_{ij} = 2u_{ij}$. This form of the free energy is intended to capture the widely observed variation of the elastic moduli in granular materials as a power law function of pressure; the stresses are given by

$$\sigma_{ij} = \frac{\partial \mathcal{F}}{\partial u_{ij}} \tag{4}$$

and the stiffness tensor $M_{ijk\ell} = \partial^2\mathcal{F}/\partial u_{ij}\partial u_{k\ell}$, and thus the elastic moduli K, μ scale as u^a. In terms of the stresses, $K, \mu \sim P^{\frac{a}{a+1}}$; Jiang and Liu take $a = 1/2$ ($K, \mu \sim P^{\frac{1}{3}}$), consistent with "Hertz contacts", for GE; we shall leave a unspecified for the time being, and investigate different choices of a below. The free energy is convex only for

$$\frac{u_s^2}{\Delta^2} < \frac{2\xi(a+2)}{5a} \tag{5}$$

implying an absence of static solutions outside this region, which, remarkably, is the Drucker-Prager form of the Coulomb yield criterion in terms of the stresses [2].

THE YIELD ANGLE

GE also defines the yield angle in terms of the material constants a and ξ. For the case of a granular layer subject to a normal stress N and shear stress T, there are two non-zero strain components, one normal (u) and one shear (γ); they are given in terms of the stresses by equation 4, from which one may obtain

$$\tan\phi = \frac{T}{N} = \frac{\frac{\gamma}{u}}{\frac{2}{5}\xi(a+2)+\frac{1}{2}a\frac{\gamma^2}{u^2}+\frac{\frac{2}{3}a^2+2a+\frac{4}{3}}{a+1}} \tag{6}$$

As there is a limit on the ratio of shear to normal strain (equation 5), substituting this critical value of the strain ratio,

$$\frac{\gamma^2}{u^2} < \frac{4\xi(a+2)}{5a} - \frac{4}{3} \tag{7}$$

into equation 6 gives the yield angle ϕ_y in terms of only the constants a and ξ:

$$\tan\phi_y = \frac{\sqrt{\frac{\xi(a+2)}{5a}-\frac{1}{3}}}{\frac{2}{5}\xi(a+2)+\frac{2}{3}} \tag{8}$$

Of course, we must also pick a value for the exponent a, and it is apparent from equation 8 that the yield angle depends on a. The Hertz theory for spheres suggests $a = 1/2$, while a large body of experimental evidence suggests $a \sim 1$ for dry, cohesionless materials (see, e. g., [8, 9]), which may be due to non-spherical contacts, an increasing number of contacts [8], or soft oxidized layers surrounding the spherical particles [10]. Consider, then, $\phi_y(\xi)$ for $a = 1/2$ and $a = 1$, shown in Figure 1. Equation 8 simplifies to

$$\phi_y = \arctan\left(\frac{\sqrt{\xi-\frac{1}{3}}}{\xi+\frac{2}{3}}\right) \tag{9}$$

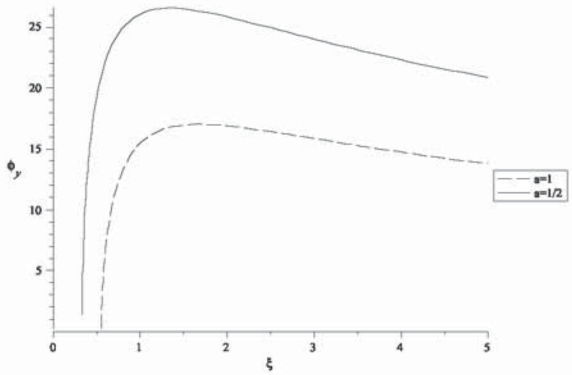

Figure 1. The yield angle ϕ_y for GE as a function of ξ. The maximum occurs at $\sim 25.5^\circ$ for $a = 1/2$ and $\sim 17^\circ$ for $a = 1$.

for $a = 1/2$ and

$$\phi_y = \arctan\left(\frac{\sqrt{\frac{3}{5}\xi - \frac{1}{3}}}{\frac{6}{5}\xi + \frac{2}{3}}\right) \tag{10}$$

for $a = 1$. The peculiar feature of these curves is that they have a maximum; for GE ($a = 1/2$) it occurs at $\xi = 4/3$, $\phi_y \approx 25.5^\circ$. So no matter what the value of the constant ξ, granular elasticity predicts a yield angle that is, at most, 25.5°. This, of course, is lower than the 30-40° observed for many materials.

While the choice of $a = 1/2$ is motivated by the Hertz theory, we have also seen that real granular materials, except at high pressures [8], generally have elastic moduli varying as $P^{1/2}$, implying $a \sim 1$. We shall subsequently refer to this case as GE-C, for "granular elasticity - cubic", since the free energy is a cubic function of the strains:

$$\mathcal{F} = \tfrac{2}{5}\xi\mathcal{A}\Delta^3 + \mathcal{A}\Delta u_s^2 \tag{11}$$

In this case, $\phi_y(\xi)$ has a similar shape, but shifts to even lower values of ϕ_y, with a maximum at $\xi = 5/3$, $\phi_y \approx 17^\circ$. The peak in the curve, as a function of a, can be determined by solving for ξ at the maximum, $\partial\phi_y/\partial\xi = 0$, with

$$\xi_{max} = \frac{5(1+2a)}{3(a+2)} \tag{12}$$

and the value of the maximum yield angle given by

$$\phi_{y,max} = \arctan\left(\frac{1}{4}\sqrt{\frac{3}{a(a+1)}}\right) \tag{13}$$

We see in Figure 2 that $\phi_{y,max}$ is a decreasing function of a. Thus, in employing GE/GE-C, one is forced to take $a \sim 1/4$, say, to allow for realistic yield angles (a maximum of $\sim 37.76^\circ$ in that case), but contrary to what

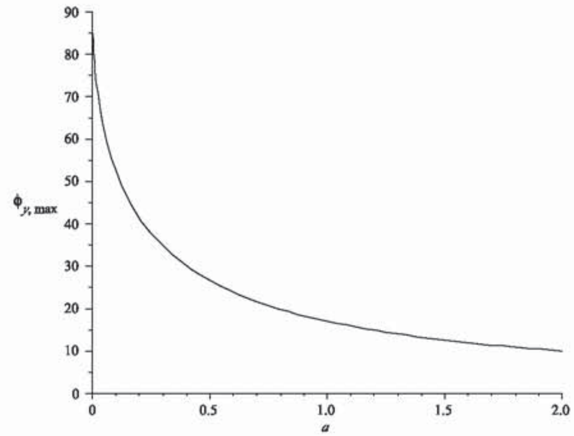

Figure 2. The maximum yield angle for GE, a decreasing function of a.

we know about the pressure dependence of the elastic moduli, or take $a \sim 1$ to match the latter, implying sand piles cannot be stable if steeper than $\sim 17^\circ$. $a = 1/2$ is something of a compromise in that regard. Below we investigate two potential generalizations in hopes of alleviating this discrepancy.

GENERALIZING THE GRANULAR ELASTICITY THEORY

Differing dependence of bulk and shear moduli on compression

Originally, Jiang and Liu considered the following more general form for the free energy [1]:

$$\mathcal{F} = \tfrac{2}{5}\xi\mathcal{A}\Delta^{b+2} + \mathcal{A}\Delta^a u_s^2 \tag{14}$$

of which granular elasticity is the special case $a = b = 1/2$. This gives the freedom to choose $a < 1/2$ in an attempt to allow higher yield angles, and $b = 1$ such that, for isotropic compression, the bulk modulus has the desired $P^{1/2}$ dependence. First, consider the effect on the stability condition, which simplifies to

$$\frac{u_s^2}{\Delta^2} \leq \frac{4\xi(b+1)(b+2)}{5a(a+1)}\Delta^{b-a} \tag{15}$$

The presence of the term Δ^{b-a} on the right hand side means the limiting stress ratio is no longer constant, but depends on the compression Δ; so the coulomb yield condition of GE is lost. Taking $a \neq b$ here will also result in the effective shear and bulk moduli having different dependence on the pressure, but both moduli usually have the same dependence on pressure (see, e.g., [9]).

Most importantly, for any anisotropic stress state (i.e. $u_s \neq 0$), this modification will not achieve the desired $P^{1/2}$ (Δ^1) dependence of the elastic moduli. While $b = 1$ ensures this relationship for isotropic stress, recall that we require $a < 1/2$ for realistic yield angles. If $u_s \neq 0$, this term will have a lower order dependence on the strains than the "bulk" term, and since the strains are small, the shear term will dominate, and the result will remain $K, \mu \sim u_{ij}^a \sim P^{\frac{a}{a+1}}$.

Incorporating dependence on the third strain invariant

In a more recent paper [7], Jiang and Liu propose a generalization of the form [8]:

$$\mathcal{F} = \Delta^{a+2} f\left(\frac{u_s^2}{\Delta^2}, \frac{u_{III}^3}{\Delta^3}\right) \tag{16}$$

where f is an arbitrary function, and u_{III}^3 is the third invariant of the (deviatoric) strain tensor,

$$u_{III}^3 \equiv u_{ij}^0 u_{jk}^0 u_{ki}^0 \tag{17}$$

For GE, $a = 1/2$ and f is given by

$$f = \mathcal{A}\left(\frac{2}{5}\xi + \frac{u_s}{\Delta^2}\right) \tag{18}$$

The simplest generalization that incorporates the third invariant is

$$f = \mathcal{A}\left(\frac{2}{5}\xi + \frac{u_s}{\Delta^2} + \zeta\frac{u_{III}^3}{\Delta^3}\right) \tag{19}$$

where ζ is another dimensionless constant; we anticipate, then, that the yield angle will be given in terms of two constants ζ and ξ, potentially resulting in a larger range of allowable values. With $a = 1$, the free energy is now given by

$$\mathcal{F} = \mathcal{A}\left(\tfrac{2}{5}\xi\Delta^3 + \Delta u_s^2 + \zeta u_{III}^3\right) \tag{20}$$

Stability requires that all eigenvalues of the stiffness matrix are positive. For the plane problem, with only two non-zero strain components, there are two eigenvalues, leading to the stability conditions

$$\frac{\gamma^2}{u^2} \leq \frac{8(-\zeta^2 + 3\zeta + 18)}{9\zeta^2} \tag{21}$$

$$\frac{\gamma^2}{u^2} \leq \frac{8(-3\zeta^2\xi - 5\zeta - 15 + 27\xi)}{9(2\zeta^2\xi + 5\zeta + 10)} \tag{22}$$

First we must establish which of these is the more stringent condition. As the value of γ^2/u^2 cannot be negative, and ξ and ζ are positive constants, we can first establish some limits on their values to ensure real solutions. The first condition does not depend on ξ at all, and it is clear that for the right hand side to be greater than zero, $\zeta < 6$. The second condition further requires that

$$-3\zeta^2\xi - 5\zeta - 15 + 27\xi > 0 \tag{23}$$

or

$$\xi(27 - 3\zeta^2) > 5\zeta + 15 \tag{24}$$

Regardless of the value of ξ, this can only be satisfied for $\zeta < 3$. Then we can also write

$$\xi > \frac{5\zeta + 15}{27 - 3\zeta^2} \tag{25}$$

In order to establish which of the two stability criteria is more severe, we shall first maximize the second with respect to ξ. Since

$$\frac{\partial}{\partial\xi}\left(\frac{8(-3\zeta^2\xi - 5\zeta - 15 + 27\xi)}{9(2\zeta^2\xi + 5\zeta + 10)}\right) = \frac{40(-\zeta^3 + 27\zeta + 54)}{9(2\zeta^2\xi + 5\zeta + 10)^2} \tag{26}$$

which is positive for $0 < \zeta < 3$, the second stability condition is an increasing function of ξ. There is no upper bound on the value of ξ, and

$$\lim_{\xi\to\infty}\left(\frac{8(-3\zeta^2\xi - 5\zeta - 15 + 27\xi)}{9(2\zeta^2\xi + 5\zeta + 10)}\right) = \frac{4(9 - \zeta^2)}{3\zeta^2} \tag{27}$$

So, having chosen ξ to make condition two as lenient as possible, we may rewrite the two conditions as follows:

$$\frac{3\zeta^2}{4}\frac{\gamma^2}{u^2} \leq \left(-\frac{2}{3}\zeta^2 + 2\zeta + 12\right) \tag{28}$$

$$\frac{3\zeta^2}{4}\frac{\gamma^2}{u^2} \leq (9 - \zeta^2) \tag{29}$$

Figure 3 shows that the second condition is more restrictive of the two, even in the limit $\xi \to \infty$. Identifying once again the ratio of shear to normal forces as the tangent of the yield angle, and substituting the stability condition (equation 22) for the ratio of shear to normal strain, we arrive at an expression for the yield angle in terms of the two constants ξ and ζ:

$$\phi_y = \arctan\left(\frac{\sqrt{\left(\zeta^2\xi + \frac{5}{2}\zeta + 5\right)\left(\frac{1}{6}\zeta - 1\right)^2(-3\zeta^2\xi - 5\zeta - 15 + 27\xi)}}{\frac{9}{5}\zeta^2\xi^2 + 3\zeta\xi + 18\xi + 2\zeta^2\xi + 5\zeta + 10 - \frac{1}{6}\zeta^3\xi - \frac{5}{9}\zeta^2}\right) \tag{30}$$

For $\zeta = 0$, this simplifies to the relationship obtained from GE-C (equation 10). Recall that GE-C has a maximum yield angle of $\sim 17^\circ$ at $\xi = 5/3$. Unfortunately, the

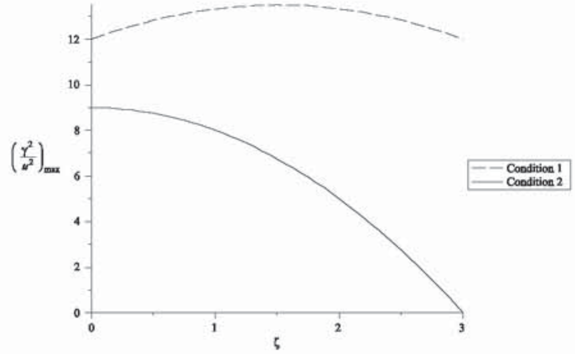

Figure 3. Maximum stable value of γ^2/u^2 for the two stability criteria, equations 28-29, in the limit $\xi \rightarrow \infty$. Stability is lost at condition two before ever reaching condition one.

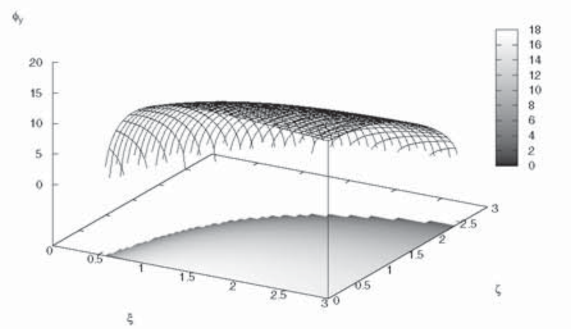

Figure 4. The yield angle ϕ_y as a function of the constants ξ and ζ. ϕ_y decreases rapidly with ζ.

preceding generalization does not allow for higher yield angles. Figure 4 shows the yield angle as a function of ξ and ζ. The familiar shape of $\phi_y(\xi)$ from GE/GE-C is apparent on the $\zeta = 0$ axis, but taking non-zero values of ζ only decreases the yield angle. The peak of the surface $\phi_y(\xi, \zeta)$ occurs at $\xi = 5/3$, $\zeta = 0$ with the maximum yield angle still approximately equal to 17°.

SUMMARY

The recently proposed granular elasticity theory has been successful in describing many aspects of granular physics, including dilatancy, stress-induced anisotropy, yield, and stress distributions in sand piles and silos [1–7]. Nevertheless, it possesses a significant quantitative discrepancy; choosing a form of the free energy that gives $K, \mu \sim P^{\frac{1}{2}}$, in accordance with experimental data, implies yield angles no larger than $\sim 17^\circ$. Here we investigate two possible generalizations of theory in hopes of relaxing this constraint; neither has proven to be successful. In particular, a form incorporating the third invariant of the strain tensor restricts the range of yield angles to even lower values. While thermodynamic instability is an intuitive description of yield, as the yield condition resulting from these forms over-constrains, it may be desirable to use a hyperelastic form that is convex everywhere [11, 12]. Such cases will be considered in a subsequent paper.

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