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Total Order Reliability in PSA: Importance of Basic Events and Systems

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Abstract: The purpose of this work is twofold. First, to formalize the properties of the total order reliability importance measure for PSA models. Second, to extend the definition of the total order importance measure to groups of basic events. This allows one to obtain the importance of systems and to address the relevance of interactions among systems.

Keywords: PSA, Importance Measures, Interactions, Total Order Reliability Importance

1. INTRODUCTION

In recent works, it has been underlined the relevance to decision makers of accounting for interactions in reliability models [Zio and Podofillini (2006); Gao et al (2007); Borgonovo (2010b)]. Zio and Podofillini (2006) extend the differential importance measure to include second order interactions. Gao et al (2007) extend the joint reliability importance to a generic order k. Do van et al (2008), extend the differential importance up to an order k (D^k). In Borgonovo (2010b) the total order reliability importance measure is introduced, that accounts for the interactions of all orders. An algorithm introduced in the same work and implemented in Borgonovo (2010c), allows one to estimate D^T at the same computational cost of the risk reduction and risk achievement worth importance measures. However, due to the recent introduction, the application of D^T to PSA models has not been studied yet.

In this work, we study the theoretical aspects of the application of D^T to PSA models. We discuss the definition of D^T first and compare its meaning to that of other importance measures. We establish a clear link between D^T and the Fussell-Vesely (FV) importance measure. In particular, we show that D^T plays, in terms of contribution to change in risk, the same role of FV in terms of contribution to risk. We show that no interactions containing groups of initiating event frequencies can be present. These results are obtained at the basic event level. Thus, the information obtained by a decision-maker is the importance of a basic event in interaction with the other basic events.

However, in several applications, it might be of interest to address the question of determining the interactions between systems. To do so, it is necessary to extend the definition of D^T from individual basic events to groups of basic events. We show that the mathematical framework as the basis of D^T allows its natural extension at the system level. We complement our discussion with numerical findings for a sample system analyzed both at the basic event and system levels.

The remainder of the paper is organized as follows. Section 2 presents the definition of D^T and its relationship to FV. Section 3 presents the theoretical findings for D^T in PSA models. Section 4 presents the definition and properties of D^T for groups at the system level. Section 5 offers conclusions and future work perspectives.

2. TOTAL ORDER RELIABILITY IMPORTANCE MEASURE

In this section, we summarize the definition and properties of D^T for reliability functions. Given any system, coherent or non-coherent, Borgonovo (2010b) shows that the system reliability function (G) is a multilinear function of the basic event probabilities ($\mathbf{x}$):

$$G(\mathbf{x})=\sum_{r=1}^{M}\sum_{m_{i_1}<m_{i_2}<\ldots<m_{i_r}}\prod_{s_1=1}^{m_1}x_{i_{s_1}}\prod_{\substack{s_2=1\\x_{i_{s_2}}\neq x_{i_{s_1}}}}^{m_2}x_{i_{s_2}}\cdot\ldots\cdot\prod_{\substack{s_r=1\\x_{i_{s_r}}\neq\ldots\neq x_{i_{s_2}}\neq x_{i_{s_1}}}}^{m_r}x_{i_{s_r}} \quad (1)$$

In eq. (1), M is the number of minimal cut sets (MCS) [for further details we refer to Borgonovo (2010c).] By the multilinearity of G(**x**), it is then possible to prove that the change in G can be exactly approximated by Taylor expansion, up to an order T. Let $\mathbf{x}^0$, $\mathbf{x}^1$ be the base case value of the basic event probabilities (**x**), then, we have:

$$\Delta G=G(\mathbf{x})-G(\mathbf{x}^0)=\sum_{i=1}^{N}B_i(\mathbf{x}^0)\cdot\Delta x_i+\sum_{k=2}^{T}\sum_{i_1<i_2,\ldots<i_k}J^k_{i_1,i_2,\ldots,i_k}(\mathbf{x}^0)\prod_{s=1}^{k}\Delta x_{i_s} \quad (2)$$

T is lower than or most equal to the number of basic events in the model.
In eq. (2), it is possible to identify the exact fraction of the change in reliability associated with basic event x_l as follows:

$$\Delta^T G_l:=B_l\Delta x_l+\sum_{k=2}^{T}\sum_{\substack{i_1<i_2,\ldots<i_k\\l\in i_1<i_2,\ldots<i_k}}J^k_{i_1,i_2,\ldots,i_k}(\mathbf{x}^0)\prod_{s=1}^{k}\Delta x_{i_s} \quad (3)$$

The total order reliability importance of x_l is then defined as:

$$D_l^T:=\frac{\Delta^T G_l}{\Delta G}=\frac{B_l\Delta x_l+\sum_{k=2}^{T}\sum_{\substack{i_1<i_2,\ldots<i_k\\l\in i_1<i_2,\ldots<i_k}}J^k_{i_1,i_2,\ldots,i_k}(\mathbf{x}^0)\prod_{s=1}^{k}\Delta x_{i_s}}{\sum_{i=1}^{N}B_i(\mathbf{x}^0)\cdot\Delta x_i+\sum_{k=2}^{T}\sum_{i_1<i_2,\ldots<i_k}J^k_{i_1,i_2,\ldots,i_k}(\mathbf{x}^0)\prod_{s=1}^{k}\Delta x_{i_s}} \quad (4)$$

In eq. (4), $J^k_{i_1,i_2,\ldots,i_k}$ is the joint reliability importance of order k of basic events $x_{i_1},x_{i_2},\ldots,x_{i_k}$.

It can be shown that if Δx tends to zero, then $D_l^T\to DIM_l^T$, where DIM_l^T is the differential importance measure [Borgonovo and Apostolakis (2001)] of basic event x_l. Also, if it is assumed $\Delta x_i=\Delta x_j$ then D_l^T becomes proportional to the Birnbaum importance (B_l) of x_l.

Eq. (4) shows that D_l^T conveys the importance of x_l in consideration of its individual effect ($B_l\,\Delta x_l$) and its interactions ($J^k_{l,i_2,\ldots,i_k}$) with all other basic event groups. In spite the fact that D_l^T contains the joint reliability importance measures of all orders related to basic even x_l, its computation does not require the calculation of all these partial derivatives. The explanation is given in Borgonovo (2010a), and is as follows. It has been shown [Sobol' (2003)] that, given any multivariate function g(x): X→R^n→R and any two points $\mathbf{x}^0$, $\mathbf{x}^1\in$X; any (finite or infinitesimal) change $\Delta g=g(\mathbf{x}^1)-g(\mathbf{x}^0)$ can be decomposed as follows:

$$\Delta g=\sum_{s=1}^{n}\sum_{i_1<i_2<\ldots<i_s}^{n}\xi_{i_1,i_2,\ldots,i_s} \quad (5)$$

Where

$$\begin{cases} \xi_i = g(x_i^1, \mathbf{x}^0_{(-i)}) - g(\mathbf{x}^0) \\ \xi_{i,j} = g(x_i^1, x_j^1, \mathbf{x}^0_{(-i,j)}) - \Delta_i g - \Delta_j g - g(x^0) \\ \dots \end{cases} \quad (6)$$

In eq. (6), ξ_i is called a finite change sensitivity index (FCSI) of order 1. ξ_i accounts for the individual contribution of x_i to the variation of g. By normalizing ξ_i, one obtains the sensitivity measures

$$\Phi_i^1 = \frac{\xi_i}{\Delta g} \quad (7)$$

Let us then consider all the terms in Δg associated with x_l. We write:

$$\xi_i^T = \sum_{s=1}^{k} \sum_{\substack{i_1 < i_2 \dots < i_s \\ i \in i_1 < i_2 \dots < i_s}} \xi_{i_1 i_2 \dots i_s} \quad (8)$$

ξ_i^T is a total order FCSI [Borgonovo (2010a)]. The corresponding normalized sensitivity measure is

$$\Phi_i^T := \frac{\xi_i^T}{\Delta g} \quad (9)$$

Φ_i^T is the fraction of the change in g associated with x_i. Let now G(**x**) be a reliability function. By results in Borgonovo (2010b) and Borgonovo (2010c), it is possible to show that:

$$\Phi_i^T = \frac{B_i'(x^0)\mathrm{d}x_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} J_{j,i}''(x^0)\mathrm{d}x_j\mathrm{d}x_i + \dots + J_{1,2,\dots,n}^{n}(x^0)\mathrm{d}x_1\mathrm{d}x_2\dots\mathrm{d}x_n}{\Delta g} = D_i^T \quad (10)$$

Eq. (10) states that the total order reliability importance measure is of x_i coincides with its total order finite change sensitivity index. This equality is enabled by the multilinearity of the reliability function [Borgonovo(2010a).] Hence, the computational shortcut utilized to compute Φ_i^T can be also utilized to estimate D_i^T. In particular, it turns out that

$$D_i^T = \frac{G(\mathbf{x}^1) - G(x_i^0, \mathbf{x}^1_{(-i)})}{\Delta G} \quad (11)$$

where $G(x^1)$ is the system unreliability with all basic event probabilities at x^1, $G(x_i;x^1_{(-i)})$ is the system unreliability with all basic event probabilities at x^1 but x_i.

Eq. (11) allows a notable reduction in computational burden. In particular, if n is the number of basic event probabilities, then n+1 model runs are necessary to obtain all basic events D^T. This cost is the same as that for obtaining the Risk Achievement Worth (RAW) or the Risk Reduction Worth (RRW) importance measures. Furthermore, by additional n+1 model runs, one can obtain the quantities

$$D_i^1 = \frac{G(x_i^1, \mathbf{x}^0_{(-i)}) - G(\mathbf{x}^0)}{\Delta G} = \frac{B_i \Delta x_i}{\Delta G} \quad (12)$$

In eq. (12), the last equality holds true by eq. (3). D_i^1 [eq. (12)] represents the individual contribution of x_i to the finite change in reliability.
Then, the difference

$$D_i^I = D^T - D_i^1 \tag{12}$$

represents the portion of the total order importance of x_l associated with interactions.
In this section, results concern reliability functions. In the next section, we discuss whether these properties hold also for PSA models.

3. PROPERTIES OF TOTAL ORDER RELIABILITY IN PSA MODELS

Since D^T has not been applied to PSA models before, in this section, we discuss its properties for PSA models.

To be able to assert that the properties of D^T hold also for PSA models, it is necessary to show that the expression of the PSA risk metric is a multilinear function. Let ϕ be the Boolean variable of the top-event, with ϕ=1, if the top-event happens. Then, letting R denote the risk metric, it is:

$$R(f_{IE}, p_{BE}) = \sum_{j=1}^{n_{IE}} f_{IEj} P(\phi = 1|IE_j) \tag{13}$$

where $P(\phi=1|IE_j)$ is the conditional probability that the top event happens given that IE_j has happended, n_{IE} the number of initiating events, and f_{IEj} the frequency of IE_j. Eq. (17) is multi-linear, since if $P(\phi=1| IE_j)$ is. Indeed, $P(\phi=1|IE_j)$ is multilinear by Theorem 1 in Borgonovo (2010a). By this, we obtain that all properties of D^T are true for PSA models. Furthermore, the following holds.

Proposition 1

1. The Birnbaum Importance of an initiating event frequency is equal to the conditional probability of the top event given the initiating event has happened;

2. The joint reliability importance of any group of initiating event frequencies is null.

3. Under the rare event approximation, the joint reliability importance of a minimal cut set equals the sum of IE frequencies involving the given minimal cut set. Also, there is no interaction of order higher than the order of the largest minimal cut set plus 1.
The proof is omitted for brevity.

4. By the multilinearity of the risk-metrics, we have:

$$D_i^T = \frac{\text{fraction of the change in risk associated with the change in } x_i}{\text{change in risk}}$$

and

$$FV_i = \frac{\text{fraction of risk associated with } x_i}{\text{risk}}$$

Thus, D^T addresses the influence of a PSA element given a change in the status quo, FV addresses the influence in respect of the status quo.
Let us ask a question: is there any condition under which these two fractions coincide?
The answer is that, in general, this is not true. As an example, consider a 2 out of 3 system, with reliability function given by:

$$y = x_1x_2 + x_1x_3 + x_2x_3 - 2x_1x_2x_3$$

.

Let x_1=0.1, x_2=0.2 and x_3=0.3. Let Δx_1=0.01, Δx_2=0.02 and Δx_3=0.03 (in other words, the components sustain a deterioration of 10% in their reliabilities). Then, it is readily shown that FV_1=0.38776, FV_2=0.69388 and FV_3=0.79592, while $D^T{}_1$=0.21163, $D^T{}_2$=0.38415, $D^T{}_3$=0.44166. Thus, the importance measures do not lead to the same numerical values. However, their interpretation is as follows: component 1 is associated with 38% of the risk (FV) and contributes to 21% of the risk change (D^T); component 2 is associated with 69% of the risk and contributes to 38% of the risk change; component 3 is associated with 80% of the risk and contributes to 44% of the risk change.
It is also readily seen that the two importance measures do not lead to the same ranking, in general. Suppose that the following changes are registered: Δx_1=0.1, Δx_2=0.02 and Δx_3=0.03. Then, the FV of the three components in the same as before. However, we have $D^T{}_1$=0.72858, $D^T{}_2$=0.14327 and $D^T{}_3$=0.17927, with component 1 becoming now the key-driver of the risk change.

However, there is a particular situation in which the two importance measures would produce the same values for any change. Suppose for the moment that the system is perfectly reliable, i.e., x_1=0, x_2=0 and x_3=0. Then, introduce changes equal to the component unreliabilities at the base case: Δx_1=0.1, Δx_2=0.2 and Δx_3=0.3 (note that these values are our previous x_1, x_2, x_3). Then, $D^T{}_1$=0.38776, $D^T{}_2$=0.69388 and $D^T{}_3$=0.79592. In other words, D^T and FV coincide when the system passes from perfectly reliable to the status quo. Thus, FV can be read as the contribution to the change in risk when the system shifts from perfectly reliable to the base case. Of course, these properties are allowed by the multilinearity of the risk metric expression.

4. TOTAL ORDER RELIABILITY FOR SYSTEMS

In several applications, the decision-maker is interested in the importance of groups of system structures and components, which are identified by multiple basic events in the PSA model. Issues in the definition of importance measures for groups were identified in Cheok et al (1998): "*there is no simple relationship between importance measures evaluated at the single component level and those evaluated at the level of a group of components, and, as a result, some of the commonly used importance measures are not realistic measures of the sensitivity of the overall risk to parameter value changes* [Cheok et al (1998); p. 213]."

We show that the mathematical structure at the basis of D^T grants one with a clear formalism to achieve this goal. Consider the set of all basic events and initiating events, **x**, partitioned in Q groups as follows

$$\underbrace{x_1\, x_2\, ..\, x_{s1}}_{\gamma_1}\ \underbrace{x_{s1+1}\, x_{s1+2}\, ..\, x_{s2}}_{\gamma_2} \dots \underbrace{x_{s_{kQ-1}+1}\, x_{s_{Q-1}+2}\, ..\, x_n}_{\gamma_Q}$$

Then, consider the fraction of ΔG [eq. (2)] associated with group i, which we write as

$$D^T_{\gamma_i} = \sum_{l\in\gamma_i} B_l \Delta x_l + \sum_{k=2}^{T} \sum_{\substack{i_1<i_2,..<i_k \\ l\in i_1<i_2,..<i_k\ \forall l\in\gamma_i}} J^k_{i_1,i_2,\dots,i_k}(\mathbf{x}^0) \prod_{s=1}^{k} \Delta x_{i_s} \tag{14}$$

where the sum is intended for all terms involving any of the basic events in group i without repetition. Consequently, we define the total order reliability importance of group i as:

$$D^T_{\gamma_i} := \frac{\Delta^T G_{\gamma_i}}{\Delta G} = \frac{\sum_{l\in\gamma_i} B_l \Delta x_l + \sum_{k=2}^{T} \sum_{\substack{i_1<i_2,..<i_k \\ l\in i_1<i_2,..<i_k\ \forall l\in\gamma_i}} J^k_{i_1,i_2,\dots,i_k}(\mathbf{x}^0) \prod_{s=1}^{k} \Delta x_{i_s}}{\Delta G} \tag{15}$$

$D^T_{\gamma_i}$ represents the fraction of the change in risk associated with group i. Note that $D^T_{\gamma_i}$ contains all the Birnbaum and joint reliability importance measures related to the basic events and/or initiating events in group i.

The question is, then, whether additional burden in the calculation of groups importance measures is generated by this definition. The answer is again found in the link between the D^T and finite change sensitivity indices. Results in Borgonovo and Peccati (2009) state that the decomposition of a function in respect of groups of independent variables is governed by the same formalism as that of individual parameters. Let us denote by $\boldsymbol{\gamma} = \{\gamma_1, \gamma_2, ..., \gamma_Q\}$ the vector of groups. Any finite change in *g* can be decomposed as follows [Borgonovo and Peccati (2009)]

$$\Delta g = g(\boldsymbol{\gamma}^1) - g(\boldsymbol{\gamma}^0) = \sum_{i=1}^{Q} \Delta_{\gamma_i} g + \sum_{i<j}^{Q} \Delta_{\gamma_i,\gamma_j} g + ... + \Delta_{\gamma_{i_1},\gamma_{i_2},...,\gamma_{i_Q}} g \tag{16}$$

where

$$\begin{cases} \Delta_{\gamma_i} g = g(\gamma^1_i; \boldsymbol{\gamma}^0_{(-i)}) - g(\boldsymbol{\gamma}^0) \\ \Delta_{\gamma_i,\gamma_j} g = g(\gamma^1_i, \gamma^1_j; \boldsymbol{\gamma}^0_{-(i,j)}) - \Delta_{\gamma_i} g - \Delta_{\gamma_j} g - g(\boldsymbol{\gamma}^0) \\ \cdots \end{cases} \tag{17}$$

Eqs. (16) and (17) are formally identical to the decomposition of Δg [eq. (5)] in terms of individual parameters. In other words, the mathematics allows one to replace the individual variables x_i with the group of variables γ_i. This also implies that all findings of the previous section hold unaltered, if one considers groups of variables instead of individual ones. Therefore, D_i^T is numerically determined by

$$D^T_{\gamma_i} = \frac{G(\boldsymbol{\gamma}^1) - G(\gamma^0_i, \boldsymbol{\gamma}^1)}{\Delta G} \tag{18}$$

where $\boldsymbol{\gamma}^1$ denotes the vector of groups fixed at $\boldsymbol{\gamma}^1$ and $(\gamma^0_i, \boldsymbol{\gamma}^1)$ denotes the point obtained by only group γ_i at the base case value, while all other groups are kept at $\boldsymbol{\gamma}^1$.

As an example, let us compute the importance for the 2 out of 3 system. Again, $x_1=0.1$, $x_2=0.2$ and $x_3=0.3$, $\Delta x_1=0.01$, $\Delta x_2=0.02$ and $\Delta x_3=0.03$. Let us consider the following groups: $\gamma_1=x_1$ and $\gamma_2=\{x_2,x_3\}$. We have $DT_{\gamma 1}$ unaltered and $DT_{\gamma 2}=0.8013$.

We note that groups are lower in number than parameters. In principle, then, this allows us to obtain the complete decomposition of the change, so that to have the quantification of the strength of interactions among systems of all orders.

4. CONCLUSIONS

In this work we have analyzed the theoretical aspects of the application of the total order reliability (DT) importance measures in PSA models. We have derived several properties and discussed in depth its relationship to the FV importance measure. The numerical aspects of the application are discussed in a next work, Smith and Borgonovo (2010).

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