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Nodal Diffusion Burnable Poison Treatment for Prismatic Reactor Cores

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Abstract –*The prismatic block version of the High Temperature Reactor (HTR) considered as a candidate Very High Temperature Reactor (VHTR) design may use burnable poison pins in locations at some corners of the fuel blocks (i.e., assembly equivalent structures). The presence of any highly absorbing materials, such as these burnable poisons, within fuel blocks for hexagonal geometry, graphite-moderated High Temperature Reactors (HTRs) causes a local inter-block flux depression that most nodal diffusion-based methods have failed to properly model or otherwise represent. The location of these burnable poisons near vertices results in an asymmetry in the morphology of the assemblies (or blocks). Hence the resulting inadequacy of traditional homogenization methods, as these methods "spread" the actually local effect of the burnable poisons throughout the assembly. Furthermore, the actual effect of the burnable poison is primarily local with influence in its immediate vicinity, which happens to include a small region within the same assembly as well as similar regions in the adjacent assemblies. Traditional homogenization methods miss this artifact entirely. This paper presents a novel method for treating the local effect of the burnable poison explicitly in the context of a modern nodal method.*

I. INTRODUCTION

The treatment of highly neutron-absorbing rods or burnable poisons (BP) within the framework of neutron diffusion theory, in particular nodal methods, has historically relied on detailed lattice physics calculations (typically using neutron transport theory) and homogenization theory for the generation of equivalent nodal diffusion group constants. The success of this approach rests on the assumption that an accurate representation of the neutron physics within a fuel assembly can be achieved even if the assembly is separated from surrounding assemblies and simplified boundary conditions are applied to the transport problem. While this assumption remains valid for reactors with a thermal spectrum and small mean-free-paths, such as in the case of Light Water Reactors (LWR), new reactor concepts and designs may challenge the current paradigm. In particular, the optically-thin graphite-moderated High Temperature Reactors (HTRs), such that proposed for the prismatic Very High Temperature Reactor (VHTR), may require a multi-level approach, which could in turn require special modeling of the burnable poisons, particularly in the case of the prismatic VHTR design. A systematic outline of the proposed approaches to modeling HTRs based on neutron diffusion theory was presented in Ref.[1]. The goal of this article is to propose and present an explicit treatment of the burnable poison rods located inside homogenized hexagonal nodes, thus incorporating the *explicit* treatment of said rods into nodal hexagonal diffusion methods.

While many derivatives of the nodal approach can be found in the literature, those based on the transverse-integration procedure are of direct relevance to the present work. In particular, we will rely on a specific implementation [2, 3, 4] of the Nodal Green's Function Method (NGFM) [5] to solve the neutron diffusion equation in hexagonal geometry. This specific approach uses net currents,

as opposed to partial currents, for interface coupling quantities between adjacent nodes. The choice of the NGFM for this work is motivated by the inherent benefit stemming from the generality and flexibility of formulation it allows in the formal solution of the ordinary differential equations (ODEs). One such ODE arises for each geometric direction as a consequence of applying the transverse-integration procedure [6] to the full 3-D neutron diffusion equation. The next paragraph describes the content and organization of this article.

Section II presents a historical perspective on the traditional homogenization approach and the steps involved in the preparation of data for subsequent nodal diffusion calculations. In addition, Section II gives an outline of the new approach and the related data preparation steps. Section III presents the physical considerations related to the in-situ neutronic behavior of the burnable poisons for the particular case of the prismatic HTR and offers modeling considerations. Finally, the neutron diffusion equation in hexagonal geometry is introduced and the application of the NGFM is outlined. Section IV presents numerical results that are used to confirm the implementation of the new method for a set of test problems. Section V concludes this article with a discussion of the ongoing work and recommendations for future work. In this article, the terms 'absorber rods' and 'burnable poison' are used interchangeably.

II. THE TRADITIONAL PARADIGM OF NODAL METHODS AND THE NEW PROPOSED ONE

Many reactor designs include the presence of burnable poison pins in certain regions of fuel assemblies or of fueled prismatic blocks in the case of the prismatic-block HTRs. The burnable poison pins are lumps of highly neutron-absorbing material that can cause local flux depressions in their immediate vicinity. Heretofore, such local flux depressions have not been properly captured when modeling the reactor with a diffusion-based nodal method. The failure to properly capture the physics of such localized heterogeneities stems from the very nature of modern nodal methods and the computational paradigm they are based upon. This paradigm is briefly summarized, next.

Modern nodal methods, and their related modern homogenization methods, are a development of the late 1970's and early to mid-1980's [7, 8]. (A more up-to-date review of homogenization methods can be found in Reference [9]). The universally used scheme in modern nodal methods follows the same set of overall steps. First, the material and geometric details at the assembly level are homogenized. The methods for carrying out this homogenization step have substantially evolved and increased in sophistication. A detailed historical retrospective can be found in Reference [10]. A present, however, the most commonly used approach is that of "Modern Equivalence Theory" [11, 12] or some variant thereof [13]. All of these approaches require the use of geometrically detailed neutron transport solutions, usually in a fine energy group structure. The solution is then used to generate spatially averaged diffusion theory data that may also be collapsed in energy. The flux solution is also then used to generate discontinuity factors.

Given the homogenized diffusion theory data, the second step in the current paradigm is to carry out the computations for the full core nodal diffusion solution of the flux and effective multiplication factor. Of course, part of the core may suffice when symmetries are taken advantage of. This second step is carried out with a variety of codes. Most modern codes are based on the transverse integration procedure that changes the full 3-D diffusion problem into a coupled set of 1-D problems [6]. The solution methods vary from polynomial approaches [6] to analytical ones [14, 15, 16, 17, 2, 18] or even analytic-function-expansion-based ones [19] (which do away with the transverse integration procedure). Since these methods were introduced in the late 1970's and early 1980's (and early 1990's for the AFEN) many others were developed that generalized the methods to geometries other than the initial Cartesian implementations or that introduced very major advances and improvements addressing computational performance efficiency and enhancements in the linear algebra algorithms and solution schemes. However, the use of polynomial, analytical or analytic function expansion solution methods has remained the universal practice. In all cases, this second step yields the full reactor flux solution and effective multiplication factor. And in all cases the methods follow the two-step approach of full-assembly-level *complete* homogenization, followed by full-core diffusion theory solution using *homogenized* data. As stated above, this approach homogenizes the burnable poisons as well as the rest of the features within the assembly and effectively distributes the effect (absorption) of the burnable poison uniformly throughout the assembly.

The new method presented in this paper incorporates a major departure from the conventional approach of modern nodal methods. While the overall two-step scheme is retained, the homogenization step is no longer required to be implemented in its entirety. That is, instead of fully homogenizing all the geometric and material

features within an assembly, most of the features are fully homogenized, while the locally confined strong absorbers such as lumped burnable poisons and control elements are *retained without homogenization*. Instead, compatible diffusion theory data are generated for them. This way a self-consistent set of diffusion theory data is generated for the background homogenized assembly domain and for the strong absorbers that are explicitly treated.

The novel approach identified above requires the development of two new computational methods. The first necessary method is one that performs the partial homogenization of the assemblies and generates the background homogenized diffusion theory data for the assembly, as mentioned above, while simultaneously generating compatible data for the localized strong absorber that is left out of the homogenization process. The second necessary method to be developed is a full-core nodal method that is capable of the explicit treatment of strong local heterogeneities. Such a method is a new development and has never been reported in the literature. Of course, in the past, nodal methods were developed that treat explicitly mild heterogeneities such as those induced by depletion or by smooth variations in local temperature [20] and that can be rendered by a low order polynomial representation with reasonable accuracy. Such older methods have even been applied to the high-fidelity modeling of realistic depletion problems [21]. The new requirement for the explicit modeling within the nodal solver of a localized strong lumped absorber departs significantly from such smooth situations.

The main focus of this paper is the development of a nodal method that incorporates the explicit treatment of strong local heterogeneities. The paper also presents a summary of the associated incomplete homogenization method. In the next Section, the underlying theoretical developments are introduced. They are followed by a description of the implementation of the method and then by the presentation of the testing of new method using a sample problem in which burnable poisons are located near some of the vertices of a prismatic fuel block. The model is limited to the minimum domain from which the effects to be investigated can be demonstrated. However, the physics incorporated in the model is representative of the physics of an actual HTR such as may be condered for the Next Generation Nuclear Plant (NGNP) or the Deep Burn HTR concept. The obtained results show that the new paradigm provides an effective means of treating BPs.

III. THEORY

The treatment of burnable poisons (BP) outlined in this article pertains specifically to the graphite-moderated prismatic HTR design. In this section, the HTR assembly design is described and certain salient features regarding the inter-assembly effects of the burnable poison are discussed. A mathematical model for the absorption behavior due to the BP is proposed and the neutron diffusion equation introduced. Finally, the NGFM is introduced and applied to the diffusion equation, thus arriving at a set of *approximate* equations for discrete variables, which can be readily solved via standard iteration techniques.

III.A. HTR Physics Considerations

The presence of burnable poisons in the prismatic HTR design is intended to hold down excess reactivity and to counteract power peaking that occurs at the vertices of the hexagonal fuel assemblies due to the so-called ‘mini-reflector’ effect, which arises from the relative higher presence of moderator material [1]. Unlike the case of LWRs, the effects from the absorption due to the BP in the prismatic HTR design is felt by neighboring assemblies, thus causing an inter-assembly flux depression. This local depression is schematically represented by a ‘radius of influence’, as highlighted in red in Figure 1.

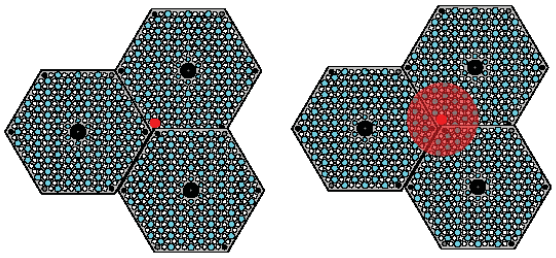

Figure 1: Radius of influence of burnable poison over neighboring assemblies.

Before accounting for the neutronic effects of the BP over neighboring fuel blocks, the neutron diffusion model must somehow account for the transport effects occurring near and inside the strongly absorbing region. This is due to the fact that diffusion theory fails to attain sufficient accuracy (with respect to transport theory) near strongly absorbing medium [22]. The approach incorporated in this work involves the application of the ‘standard’ flux-weighted homogenization, which conserves the block-wise (assembly) reaction rates, and the addition of a local absorber effective absorption. The former is used in the block-wise neutron balance to conserve the average reaction

rate, while the latter is used in the one-dimensional solution to simulate the effect of the strong absorber. This use alters the net currents and enforces an accounting for the "neighboring" effect of the BP. In the next section we discuss the mathematical formulation for the burnable poison treatment, the neutron diffusion equation, and introduce the NGFM.

III.B. Mathematical Developments

In this section we introduce a mathematical model for the absorption due to a burnable poison within a diffusive domain. Once the model is introduced, the neutron diffusion equation is presented. This presentation will include a model for the burnable poison that can readily be solved by the NGFM.

III.C. Idealized Model of Burnable Poisons

Consider the integrated reaction rate, R_a^p, due to absorption occurring inside a finite-dimensional sub-domain of volume V^p, henceforth referred to as a burnable poison (*BP*), surrounded by an infinitely homogenous material containing a time-independent distributed source which emits mono-energetic neutrons,

$$R_a^p = \int_{V^p} \Sigma_a\left(\vec{r}\right)\varphi\left(\vec{r}\right)dV \tag{1}$$

where $\varphi\left(\vec{r}\right)$ represents the neutron flux as a function of space, $\Sigma_a\left(\vec{r}\right)$ is the macroscopic absorption cross-section for the whole domain, $\vec{r} \in \mathbb{R}^3$, and all other notation, unless otherwise noted, is standard. Under the assumption that the macroscopic absorption cross-section inside the burnable poison is constant, and defining the average flux over the absorbing region in the following manner,

$$\overline{\varphi}^p \equiv \frac{1}{V^p}\int_{V^p} \varphi\left(\vec{r}\right)dV \tag{2}$$

we obtain the reaction rate due to absorption over the burnable poison in terms of the average flux,

$$R_a^p = \Sigma_a^p V^p \overline{\varphi}^p \tag{3}$$

where Σ_a^p is the constant macroscopic absorption cross-section inside the burnable poison. Any proposed equivalent model for the burnable poison absorption should at least conserve the integrated reaction rate over the sub-domain. Next, consider Eq. (1) and assume space dependence for the macroscopic absorption rate inside the burnable poison region. In particular, let us define the space-dependent absorption cross-section as follows

$$\Sigma_a^p\left(\vec{r}\right) = \delta\Sigma_a^p\left(\vec{r}-\vec{r}\,'\right) = \begin{cases} +\infty, \vec{r} = \vec{r}\,' \\ 0, \vec{r} \neq \vec{r}\,' \end{cases} \tag{4}$$

This representation of the absorption cross-section is clearly a mathematical idealization, since the actual macroscopic cross-section is constant over the burnable absorber. This model for the absorption cross-section will be justified by certain advantages that it brings to the NGFM approach, as shown later in this article. It is clear that, according to Eq. (4), the absorption cross-section behaves as a Dirac delta function (more accurately, a generalized Dirac product), and that the definition presented in this article is merely heuristic. Rigorously, the Dirac delta function is a generalized function which, according to the theory of distributions, is not a function itself but a linear functional on the space of test functions that are smooth over the domain and have compact support [23]. The proposed model for the absorption cross-section for the burnable poison, which 'collapses' the region into a line absorber, can be written as

$$\Sigma_a^p\left(\vec{r}\right) = \tilde{\Sigma}_a^p\delta\left(x-x'\right)\delta\left(y-y'\right) \tag{5}$$

where $\left(x', y'\right)$ indicates the location of the burnable poison, which will be assumed to be constant along the *z*-direction, $\delta\left(X-X'\right), X \in \left(x, y\right)$ is a *symbolic function* of the delta distribution [24], and $\tilde{\Sigma}_a^p$ is an *effective* absorption reaction rate that has yet to be determined. In light of Eq. (5), the integrated absorption reaction rate is

$$\int_{V^{BP}} \Sigma_a^p\left(\vec{r}\right)\varphi\left(\vec{r}\right)dV = \tilde{\Sigma}_a^p h^p \overline{\varphi}\left(x', y'\right) \tag{6}$$

where h^p is the height of the line absorber or collapsed burnable poison and $\overline{\varphi}\left(x', y'\right)$ is the *z*-direction averaged scalar flux evaluated at the location of the burnable poison. In order to conserve the integral reaction rate, we must enforce the following condition on the burnable poison model

$$\tilde{\Sigma}_a^p h^p \bar{\varphi}\left(x', y'\right) = \Sigma_a^p V^p \bar{\varphi}^p \quad (7)$$

Note that Eq. (7) implies that $\tilde{\Sigma}_a^p$ must be determined such that the actual absorption reaction rate (right side of the equation) should be recovered when $\tilde{\Sigma}_a^p$ is multiplied by the flux at the location of the burnable poison. In order to correctly obtain $\bar{\varphi}^p$, an auxiliary transport calculation must be performed and incorporated into Eq. (7). In addition, there is no *a priori* knowledge of $\bar{\varphi}\left(x', y'\right)$, since that would require the solution of the diffusion problem in order to compute the effective absorption cross-section. In fact, Eq. (7) may define a method similar to the *superhomogenisation* technique (SPH) [10, 25,] for generating cross-sections, whence nonlinear iterations are required in order to obtain the exact effective cross-section.

As a first approximation, the strategy in this work involves the use of the assembly-averaged flux from the lattice transport calculation to recover the transport reaction rate due to the burnable poison. This choice is justified by the fact that the nodal diffusion calculation ought to produce an assembly-averaged flux that is close to the assembly-averaged flux obtained from the heterogeneous transport lattice calculation.

III.D. Nodal Green's Function Method in Hexagonal Geometry

In this section we introduce the NGFM as previously developed for, and applied to, hexagonal geometry neutron diffusion in Reference [4]. Consider the steady-state multi-group, multi-dimensional neutron diffusion equation for a homogenous hexagonal node V^k

$$-D_g^k \vec{\nabla}^2 \varphi_g^k\left(\vec{r}\right) + \Sigma_{r,g}^k\left(\vec{r}\right)\varphi_g^k\left(\vec{r}\right) = Q_g^k\left(\vec{r}\right) \quad (8)$$

where

$$Q_g^k\left(\vec{r}\right) \equiv F\varphi_{g'}^k\left(\vec{r}\right) \quad (9)$$

$$F \equiv \left(\frac{\chi_g}{k}\sum_{g'=1}^{G}\upsilon\Sigma_{f,g'}^k + \sum_{g'\neq g}\Sigma_{s,gg'}^k\right) \quad (10)$$

where D_g^k is the direction-independent diffusion coefficient, and $\Sigma_{r,g}^k\left(\vec{r}\right)$, $\upsilon\Sigma_{f,g'}^k$, and $\Sigma_{s,gg'}^k$ are the removal, fission, and scattering macroscopic cross-sections, respectively. In addition, χ_g is the fission spectrum and k is the effective multiplication factor. Note that the removal cross-section is space-dependent within the node and reduces to the absorption cross-section for the particular case of $G = 1$. The diffusion equation is subject to general albedo boundary conditions at the edge of the domain.

The first step in deriving a nodal scheme is to form the nodal balance equation. The nodal balance equation is obtained by integrating Eq. (8) over the homogenous volume V^k. The nodal coordinate system for the hexagonal boundary in Cartesian coordinates is shown in Figure 2. The node is defined by

$$V^k : \left(x, y, z\right),\ x \in [-h, +h],$$
$$y \in [-y_s(x), +y_s(x)],\ z \in [-a, +a] \quad (11)$$

where

$$y_s\left(x\right) \equiv \frac{1}{\sqrt{3}}\left(2h - |x|\right) \quad (12)$$

and $2h$ is the lattice pitch and $2a$ is the node height.

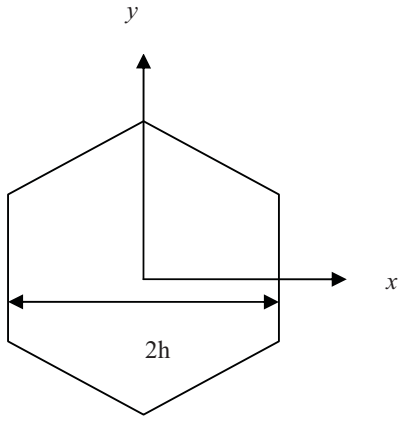

Fig. 2: Cartesian coordinate system for a hexagonal node.

We begin by operating on Eq. (8) with $\frac{1}{V^k}\int_{V^k} dV$ and define the following quantities

$$\bar{\varphi}_g^k \equiv \frac{1}{V^k}\int_{V^k}\varphi_g^k\left(\vec{r}\right)dV \tag{13}$$

$$\bar{Q}_g^k \equiv \frac{1}{V^k}\int_{V^k}Q_g^k\left(\vec{r}\right)dV \tag{14}$$

and applying Green's theorem to the leakage term yields

$$\frac{1}{V^k}\int_{V^k}dV\left(-D_g^k\vec{\nabla}^2\varphi_g^k\left(\vec{r}\right)\right)= \sum_{i=1}^{8}\frac{1}{V^k}\int_{A_i^k}dA_i\left(-D_g^k\hat{n}_i\cdot\vec{\nabla}\varphi_g^k\left(\vec{r}\right)\right) \tag{15}$$

where $\hat{n}_i$ is one of the normal vectors in each of the eight directions of the surface leakage $\left(\pm u,\pm v,\pm w,\pm z\right)$. We can define the surface averaged leakage for Eq. (15) as

$$L_{gx}^k \equiv J_{gx}^k\left(h\right)-J_{gx}^k\left(-h\right), x\in\left(u,v,w\right) \tag{16}$$

$$L_{gz}^k \equiv J_{gz}^k\left(a\right)-J_{gz}^k\left(-a\right) \tag{17}$$

For an $x-$direction, the surface averaged components of the net current is given by

$$J_{gx}^k\left(\pm h\right)\equiv\left[\frac{1}{2a}\int_{-a}^{a}dz\frac{1}{2y_s\left(x\right)}\int_{-y_s(x)}^{y_s(x)}dy-D_g^k\frac{\partial\varphi_g^k}{\partial x}\right]_{x=\pm h} \tag{18}$$

and in the $z-$direction,

$$J_{gz}^k\left(\pm a\right)\equiv\left[\frac{1}{2h}\int_{-h}^{h}dx\frac{1}{2y_s\left(x\right)}\int_{-y_s(x)}^{y_s(x)}dy-D_g^k\frac{\partial\varphi_g^k}{\partial z}\right]_{z=\pm a} \tag{19}$$

Note that the removal term on the left hand side of Eq. (8) is space-dependent and involves $\left(P+1\right)$ terms, where P denotes the total number of burnable poison rods inside node k and b denotes the removal due to the homogenized 'background' material, i.e. $\Sigma_{r,g}^k\left(\vec{r}\right)=\Sigma_{r,g}^{k,b}\left(\vec{r}\right)+\sum_{p=1}^{P}\Sigma_{r,g}^{k,p}\left(\vec{r}\right)$. If we operate over this removal term in the diffusion equation with $\frac{1}{V^k}\int_{V^k}dV$, we obtain

$$\frac{1}{V^k}\int_{V^k}\Sigma_{r,g}^k\left(\vec{r}\right)\varphi_g^k\left(\vec{r}\right)dV= \left(\frac{V^b}{V^k}\right)\Sigma_{r,g}^{k,b}\bar{\varphi}_g^{k,b}+\sum_{p=1}^{P}\left(\frac{V^p}{V^k}\right)\Sigma_{r,g}^{k,p}\bar{\varphi}_g^{k,p} \tag{20}$$

where $\bar{\varphi}_g^{k,b}$ and $\bar{\varphi}_g^{k,p}$ denote the spatially-averaged flux over the 'background' and 'poison' regions, respectively. Note that no approximation has been introduced so far in the derivation. In fact, it is not even necessary to separate the 'background' and 'poison' regions if we define a homogenized removal cross-section such as

$$\Sigma_{r,g}^k \equiv \frac{1}{V^k}\int_{V^k}\Sigma_{r,g}^k\left(\vec{r}\right)\varphi_g^k\left(\vec{r}\right)dV/\bar{\varphi}_g^k= \left(\frac{V^b\bar{\varphi}_g^{k,b}}{V^k\bar{\varphi}_g^k}\right)\Sigma_{r,g}^{k,b}+\sum_{p=1}^{P}\left(\frac{V^p\bar{\varphi}_g^{k,p}}{V^k\bar{\varphi}_g^k}\right)\Sigma_{r,g}^{k,p} \tag{21}$$

Eq. (21) corresponds to the standard flux-weighted homogenization technique; hence there is no difference between the traditional homogenization and the burnable poison treatment with respect to the balance equation. This is expected, since the balance equation must be enforced regardless of the nature or morphology of the particular heterogeneity introduced into the node, as long as the homogenized cross-sections are defined consistently.

Finally, by using the definitions expressed by Eqs. (13) through (21) , we arrive at the neutron balance equation in hexagonal geometry

$$\frac{1}{2a}L_{gz}^k+\frac{1}{3h}\left[L_{gu}^k+L_{gv}^k+L_{gw}^k\right]+\Sigma_{r,g}^k\bar{\varphi}_g^k=\bar{Q}_g^k \tag{22}$$

which requires the evaluation of the surface net currents in order to evaluate the net leakage terms. These net currents can be derived through a Green's function solution of the transverse averaged diffusion equation. (The same is true if partial currents were to be used.)

The fundamental procedure in nodal methods involves integrations of the n dimensional neutron diffusion equation over $n-1$ directions transverse in each coordinate direction [6]. The resulting set of n-coupled ordinary differential equations is approximated using techniques appropriate for the solution of the one-dimensional diffusion equation. Additional approximations to the transverse leakage terms, which couple the one-dimensional equations, are also required. Let us transverse-integrate Eq. (8)

in the $y-$ and $z-$directions (where we have expanded the Laplacian operator),

$$\int_{-a}^{a} dz \int_{-y_s(x)}^{y_s(x)} dy \left[\begin{array}{l} -D_g^k \left(\frac{\partial^2 \varphi_g^k}{\partial x^2} + \frac{\partial^2 \varphi_g^k}{\partial y^2} + \frac{\partial^2 \varphi_g^k}{\partial z^2} \right) + \\ \Sigma_{r,g}^{k,b}(\vec{r}) \varphi_g^k(\vec{r}) = \tilde{Q}_g^k(\vec{r}) \end{array} \right] \tag{23}$$

where the modified source on the right hand side is defined as

$$\tilde{Q}_g^k(\vec{r}) = Q_g^k(\vec{r}) - \sum_{p=1}^{P} \Sigma_{r,g}^{k,p}(\vec{r}) \varphi_g^k(\vec{r}) \tag{24}$$

Note that Eqs. (23) and (24) include space-dependent removal cross-sections on both sides of the equations. The first approximation introduced in this methodology assumes that the local absorber removal can be modeled in accordance with Eq. (5), hence

$$\Sigma_{r,g}^k(\vec{r}) \cong \tilde{\Sigma}_{r,g}^{k,b} + \sum_{p=1}^{P} \tilde{\Sigma}_{r,g}^{k,p} \delta(x-x') \delta(y-y') \tag{25a}$$

where

$$\tilde{\Sigma}_{r,g}^{k,b} = \left(\frac{V^b \overline{\varphi}_g^{k,b}}{V^k \overline{\varphi}_g^k} \right) \Sigma_{r,g}^{k,b} \tag{25b}$$

$$\tilde{\Sigma}_{r,g}^{k,p} = \left(\frac{V^p \overline{\varphi}_g^{k,p}}{V^k \overline{\varphi}_g^k} \right) \Sigma_{r,g}^{k,p} \tag{25c}$$

In order to transverse-integrate, we must apply Leibniz' rule multiple times, thus obtaining a relationship for the $x-$direction derivative, such that

$$\begin{aligned} &\int_{-a}^{a} dz \int_{-y_s(x)}^{y_s(x)} dy - D_g^k \frac{\partial^2 \varphi_g^k}{\partial x^2} = -D_g^k \frac{d^2}{dx^2} \varphi_{gx}^{k,yz}(x) + \\ &D_g^k y_s'(x) \left[\left. \frac{\partial \varphi_g^{k,z}}{\partial x} \right|_{y_s(x)} + \left. \frac{\partial \varphi_g^{k,z}}{\partial x} \right|_{-y_s(x)} \right] + \\ &D_g^k y_s''(x) \left[\varphi_g^{k,z}(x, y_s(x)) + \varphi_g^{k,z}(x, -y_s(x)) \right] + \\ &D_g^k y_s'(x) \left[\begin{array}{l} \frac{d}{dx} \varphi_g^{k,z}(x, y_s(x)) + \\ \frac{d}{dx} \varphi_g^{k,z}(x, -y_s(x)) \end{array} \right] \end{aligned} \tag{26}$$

where $\varphi_{gx}^{k,yz}(x)$ is the transverse-integrated (instead of *averaged*) one-dimensional flux. In a similar fashion, $\varphi_g^{k,z}(x, \pm y_s(x))$ and $\left. \frac{\partial \varphi_g^{k,z}}{\partial x} \right|_{\pm y_s(x)}$ represent the $z-$integrated flux and $x-$direction, $z-$integrated derivative evaluated along the sides of the hexagonal prism.

The formally transverse-integrated diffusion equation, Eq. (23), becomes

$$\begin{aligned} &-D_g^k \frac{d^2}{dx^2} \varphi_{gx}^{k,yz}(x) + \tilde{\Sigma}_{r,g}^{k,b} \varphi_{gx}^{k,yz}(x) = \\ &\tilde{Q}_{gx}^{k,yz}(x) - L_{gx}^{k,yz}(x) \end{aligned} \tag{27}$$

Note that we wish to arrive at the transverse-*averaged* one-dimensional diffusion equation. The transverse-averaged one-dimensional flux is defined as,

$$\begin{aligned} &\varphi_{gx}^k(x) \equiv \frac{1}{2a} \frac{1}{2y_s(x)} \varphi_{gx}^{k,yz}(x) = \\ &\frac{1}{2a} \int_{-a}^{a} dz \frac{1}{2y_s(x)} \int_{-y_s(x)}^{y_s(x)} dy \varphi_g^k(\vec{r}) \end{aligned} \tag{28}$$

This definition can be solved for $\varphi_{gx}^{k,yz}(x)$ and substituted back into Eq. (27) to obtain the following relationship

$$-D_g^k \frac{d^2}{dx^2} \varphi_{gx}^k(x) + \tilde{\Sigma}_{r,g}^{k,b} \varphi_{gx}^k(x) = \tilde{Q}_{gx}^k(x) - L_{gx}^k(x) \tag{29}$$

where we have defined the transverse-averaged leakage as follows

$$L_{gx}^{k}\left(x\right)\equiv\frac{1}{\sqrt{3}y_{s}\left(x\right)}\left[\begin{array}{l}\overline{J}_{g}^{k,z}\left(x,y_{s}\left(x\right)\right)-\\ \overline{J}_{g}^{k,z}\left(x,-y_{s}\left(x\right)\right)\end{array}\right]+$$
$$\frac{1}{2a}\left[\overline{J}_{g}^{k,y}\left(x,a\right)-\overline{J}_{g}^{k,y}\left(x,-a\right)\right]+$$
$$D_{g}^{k}\frac{y_{s}^{'}\left(x\right)}{2y_{s}\left(x\right)}\left[\begin{array}{l}\frac{d}{dx}\overline{\varphi}_{g}^{k,z}\left(x,y_{s}\left(x\right)\right)+\\ \frac{d}{dx}\overline{\varphi}_{g}^{k,z}\left(x,-y_{s}\left(x\right)\right)-\\ 4\frac{d}{dx}\varphi_{gx}^{k}\left(x\right)\end{array}\right]+$$
$$D_{g}^{k}\frac{y_{s}^{''}\left(x\right)}{2y_{s}\left(x\right)}\left[\begin{array}{l}\overline{\varphi}_{g}^{k,z}\left(x,y_{s}\left(x\right)\right)+\\ \overline{\varphi}_{g}^{k,z}\left(x,-y_{s}\left(x\right)\right)-2\varphi_{gx}^{k}\left(x\right)\end{array}\right] \quad (30)$$

We solve Eqs. (22) and (29) by setting x equal to each of the $\left\{u,v,w\right\}$ directions, effectively rotating the hexagonal plane coordinate system three times. In order to solve for the transverse-averaged leakage, the discontinuous terms that arise from the transverse-averaging procedure must be approximated. The approach developed in Reference [4]has been shown to be superior and involves accounting for these terms by imposing rigorously the nodal balance upon integration of the solution $\varphi_{gx}^{k}\left(x\right)$. The terms in the transverse-averaged leakage that involve the net currents are assumed to be polynomials and the standard three-node quadratic leakage approximation is applied [7].

Formally, Eq. (29) can be solved by noting that the transverse-averaged diffusion equation is an ordinary differential equation with a particular inhomogeneous and a homogeneous solution. There are many standard techniques for solving an elliptic ordinary differential equation, but we specifically choose the Green's function approach because of its generality, since it allows for source functions beyond just polynomials (including the delta distributions of our method). The formal solution to Eq. (29) is

$$\varphi_{gx}^{k}\left(x\right)=\int_{-h}^{h}dx_{o}\left[G_{gx}^{k}\left(x\,|\,x_{o}\right)\tilde{S}_{gx}^{k}\left(x_{o}\right)\right]+$$
$$\left[G_{gx}^{k}\left(x\,|\,x_{o}\right)D_{g}^{k}\frac{d}{dx_{o}}\varphi_{gx}^{k}\left(x_{o}\right)\right]_{x_{o}=-h}^{x_{o}=h}-\quad (31)$$
$$\left[\varphi_{gx}^{k}\left(x_{o}\right)D_{g}^{k}\frac{d}{dx_{o}}G_{gx}^{k}\left(x\,|\,x_{o}\right)\right]_{x_{o}=-h}^{x_{o}=h}$$

where the effective modified source term is defined as

$$\tilde{S}_{gx}^{k}\left(x\right)=\tilde{Q}_{gx}^{k}\left(x\right)-L_{gx}^{k}\left(x\right) \quad (32)$$

It is worthwhile to note at this point that different families of the NGFM can be derived depending on the boundary conditions applied to the Green's function. In particular, the net current NGFM is obtained by applying Neumann boundary conditions, thus yielding the solution to Eq. (29)

$$\varphi_{gx}^{k}\left(x\right)=\int_{-h}^{h}dx_{o}\left[G_{gx}^{k}\left(x\,|\,x_{o}\right)S_{gx}^{k}\left(x_{o}\right)\right]-$$
$$\left[G_{gx}^{k}\left(x\,|\,x_{o}\right)J_{gx}^{k}\left(x_{o}\right)\right]_{x_{o}=-h}^{x_{o}=h} \quad (33)$$

The physical coupling between two adjacent nodes is introduced into the formalism by requiring the one-dimensional Green's function solution transverse-averaged flux at a node boundary to be equal to the same from the neighboring node (within the constraint imposed using discontinuity factors per the prescription of modern equivalence theory), i.e.,

$$f_{gx}^{k-1}\left(h\right)\varphi_{gx}^{k-1}\left(h\right)=f_{gx}^{k}\left(-h\right)\varphi_{gx}^{k}\left(-h\right) \quad (34)$$

where $f_{gx}^{k-1}\left(h\right)$ and $f_{gx}^{k}\left(-h\right)$ are the discontinuity factors derived from equivalence or homogenization theory for nodes $k-1$ and k. The second spatial coupling between nodes requires the net currents to be equal to each other on both sides of the interface, i.e. $J_{gx}^{k-1}\left(h\right)=J_{gx}^{k}\left(-h\right)$ and $J_{gx}^{k}\left(h\right)=J_{gx}^{k+1}\left(-h\right)$. This approach gives rise to the following expression

$$-B_{gx}^{k-1}J_{gx}^{k-1}\left(-h\right)+B_{gx}^{k-1,k}J_{gx}^{k}\left(-h\right)-$$
$$B_{gx}^{k}J_{gx}^{k+1}\left(-h\right)=R_{gx}^{k-1}\left(h\right)-R_{gx}^{k}\left(-h\right) \quad (35)$$

In matrix notation, this equation can be written as

$$\underline{\underline{B}}_{gx} \cdot \vec{J}_{gx} = \vec{R}_{gx} \tag{36}$$

where the matrix $\underline{\underline{B}}_{gx}$ is a tri-diagonal matrix that couples inter-nodal net currents in each of the $\{u, v, w\}$ directions, the vector $\vec{J}_{gx}$ represent the net currents that are to solved for in the linear system, and $\vec{R}_{gx}$ is the right hand side source vector that is pre-computed before each iteration. Also, note that the matrix $\underline{\underline{B}}_{gx}$ does not contain any information other than geometric and material data, thus it can be pre-computed and factorized using LU decomposition in order to efficiently invert the linear system.

In order to solve the linear system it is necessary to have explicit representation of the source terms. Since the effective modified source term $\tilde{S}^k_{gx}(x)$ must be specified explicitly in order to evaluate the right hand side of the linear system, we must expand the transverse-averaged source and transverse-leakage into some set of orthonormal functions and explicitly evaluate the burnable poison absorption term. Let us first define the effective transverse-averaged source in the following manner,

$$S^k_{gx}(x) = Q^k_{gx}(x) - L^k_{gx}(x) \ , \tag{37}$$

where we have separated the burnable poison term from the modified effective source such that

$$\tilde{S}^k_{gx}(x) = S^k_{gx}(x) - \sum_{p=1}^{P} \frac{1}{2a} \int_{-a}^{a} dz \frac{1}{2y_s(x)} \int_{-y_s(x)}^{y_s(x)} dy \Sigma^{k,p}_{r,g}(\vec{r}) \varphi^k_g(\vec{r}) \tag{38}$$

Substituting Eq. (5) into Eq. (38) yields the following expression for the modified effective source

$$\frac{1}{2a} \int_{-a}^{a} dz \frac{1}{2y_s(x)} \int_{-y_s(x)}^{y_s(x)} dy \Sigma^{k,p}_{r,g}(\vec{r}) \varphi^k_g(\vec{r}) = \frac{\delta(x - x^p)}{2y_s(x)} \tilde{\Sigma}^{k,p}_{r,g} \overline{\varphi}^k_g(x, y^p) \tag{39}$$

Substituting Eq. (39) into Eq. (33) and evaluating the 'sink' term arising from the presence of burnable poisons yields

$$\int_{-h}^{h} dx_o \left[G^k_{gx}(x \mid x_o) \frac{\delta(x_o - x^p)}{2y_s(x_o)} \tilde{\Sigma}^{k,p}_{r,g} \overline{\varphi}^k_g(x_o, y^p) \right] = \frac{G^k_{gx}(x \mid x^p)}{2y_s(x^p)} \tilde{\Sigma}^{k,p}_{r,g} \overline{\varphi}^k_g(x^p, y^p) \tag{40}$$

The Green's function solution to the one-dimensional transverse-averaged diffusion equation becomes

$$\varphi^k_{gx}(x) = \sum_{l=0}^{2} S^k_{gxl} G^k_{gl}(x) - \sum_{p=1}^{P} G^{k,p}_g(x) \tilde{\Sigma}^{k,p}_{r,g} \overline{\varphi}^k_g(x^p, y^p) - \left[G^k_{gx}(x \mid x_o) J^k_{gx}(x_o) \right]_{x_o=-h}^{x_o=h} \tag{41}$$

Where, for convenience, we have defined the following expression

$$G^{k,p}_g(x) = \frac{G^k_{gx}(x \mid x^p)}{2y_s(x^p)} \tag{42}$$

In order to account for the fact that the discontinuous terms arising from the transverse-averaging procedure over the hexagonal node were 'dropped', we impose the balance condition upon Eq. (41) by requiring that the zero-moment over the *x*-direction be equal to the nodal averaged flux. The zero-moment of Eq. (42) is

$$\varphi^k_{gx0} = \sum_{l=0}^{2} S^k_{gxl} G^k_{g0l} - \sum_{p=1}^{P} G^{k,p}_{g0} \tilde{\Sigma}^{k,p}_{r,g} \overline{\varphi}^k_g(x^p, y^p) - \left[\overline{G}^k_{gx0}(x_o) J^k_{gx}(x_o) \right]_{x_o=-h}^{x_o=h} \tag{43}$$

Since we wish to impose the condition $\varphi^k_{gx0} = \overline{\varphi}^k_g$, it is necessary to compensate for, or cancel out, the other non-orthogonal terms. Following Reference [**Error! Bookmark not defined.**], we choose the $l = 0$ contribution to the source. Note that the expressions derived above not only will effectively change the $\underline{\underline{B}}_{gx}$ matrix so that the resulting currents are consistent with the one-dimensional solution, but they will also give rise to extra terms that modify the right side source $\vec{R}_{gx}$ due to the presence of burnable poisons.

The standard approach for solving the set of NGFM equations involves the standard inner/outer iterations in which an initial guess for the flux solution is (i.e., a set of spatial moments for each direction). The effective multiplication factor is also assumed to take some initial guess value. The inner iterations involve a directional sweep followed by an update of the transverse-leakage terms. Once the net currents for all dimensions are updated, the moments of the direction flux solution are updated and the upscattering/downscattering contributions are computed. Each outer iteration ends when all the energy groups have been solved for and the source and effective multiplication factor are updated.

IV. IMPLEMENTATION

The HEXPEDITE implementation of the NGFM in hexagonal geometry includes standard diffusion solver options such as two- and three-dimensional geometry modeling, true multi-group capabilities, upscattering treatment, power iterations for the eigenvalue search accelerated by source extrapolation (based on dominance ratio estimates, see Ref. [4]). The implementation of the burnable poison treatment required the incorporation of modifications in the representation of the effective source within the HEXPEDITE code and, outside the HEXPEDITE code, the generation of additional data that pertain to the flux-weighted homogenized removal cross-section for the burnable poison and background material. The generation of these data is discussed in the next section.

V. GENERATION OF PARTIALLY HOMOGENIZED DATA

In Section III a particular model for the space-dependent removal cross-section was introduced and relevant definitions were established. It is worthwhile to revisit the assumptions made in the burnable poison treatment from a different perspective. In particular, we are interested in discussing the generation of homogenized data in the context of the NGFM.

Recall that the neutron balance equation over a homogenized node remains identically the same as in previous nodal formulations. Hence, all diffusion theory homogenized cross-section must be generated via the following relation

$$\Sigma_{x,g}^{k} \equiv \frac{\int_{V^k} \Sigma_{x,g}^{k}\left(\vec{r}\right)\varphi_{g}^{k}\left(\vec{r}\right)dV}{\int_{V^k} \varphi_{g}^{k}\left(\vec{r}\right)dV} \qquad (44)$$

However, note that the numerator in Eq. (44) may be divided into multiple regions, due to the linearity of the integral operator, while remaining consistent with the $\Sigma_{x,g}^{k}$ definition. Thus, if we generate two sets of material and group dependent removal cross-sections for a burnable poison and background material, and weight these according to Eq. (44), these can be linearly combined to reconstruct a single homogenized cross-section to be used in the balance equation. Additionally, exactly the same homogenized removal cross-section may be used to evaluate the one-dimensional solutions (which give rise to a net current response matrix). The method just summarized was implemented though the adapted use of the HELIOS code and appropriate data were generated for the sample problem presented in the next section.

VI. NUMERICAL RESULTS

In order to verify the implementation of the new method within the HEXPEDITE code, series of numerical results were obtained for a set of manufactured problems for which the exact solution is known. These manufactured solution problems allowed the desired verification and are not discussed further in this paper.

In order to test the new methodology, a simplified case was modeled with HEXPEDITE. The model was used to verify that the expected neutron physics were effectively reproduced by the modified HEXPEDITE code. A two-dimensional mini-reactor core was modeled consisting of seven hexagonal assemblies with 36 cm flat-to-flat dimensions. In addition, a set of mono-energetic cross-sections were devised in order to obtain a slightly super-critical system. Vacuum boundary conditions were imposed by introducing a zero incoming partial currents and a single burnable poison introduced at the center fuel block. Note that the removal cross-section of this single burnable poison "pin" was made large in order to demonstrate the desired effect unequivocally.

First, an eigenvalue calculation was performed in order to obtain an 'uncontrolled' case, as shown in Figure 3, below. Due the vacuum boundary conditions, the leakage effect causes the power distribution to be lower in the peripheral assemblies than in the center assembly. Once the burnable poison is inserted into the center fuel block, the power distribution there becomes lower relative to the surrounding fuel assemblies (see Figure 4). Next, the burnable poison position is changed from the center to the 'North' and subsequently 'South' (Figures 5 and 6). It is evident from the results that the peak relative power distribution is opposite (and symmetric) the location of the burnable poison. This

is the effect that is expected from the neutronic behavior of the burnable poison when inserted near the vertices of the hexagonal block. Additional results are shown for six positions (Figures 7 through 12), which further confirm the expected behavior of the burnable poison and the correct treatment via the methodology presented in this article.

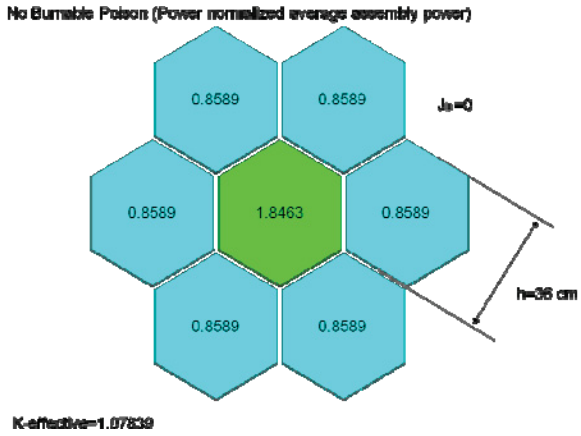

Fig. 3: Simplified mini-core effective multiplication factor and power distribution without BP.

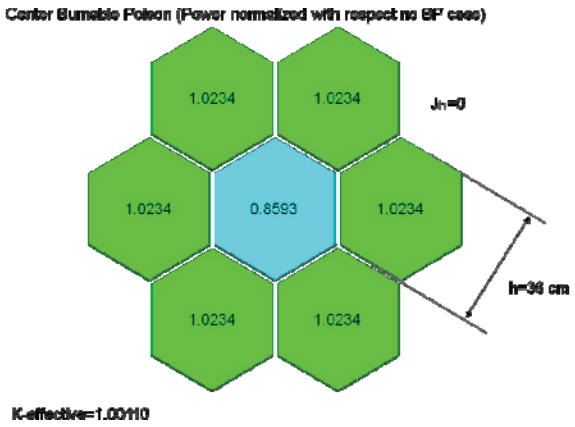

Fig. 4: Simplified mini-core effective multiplication factor and power distribution with center BP.

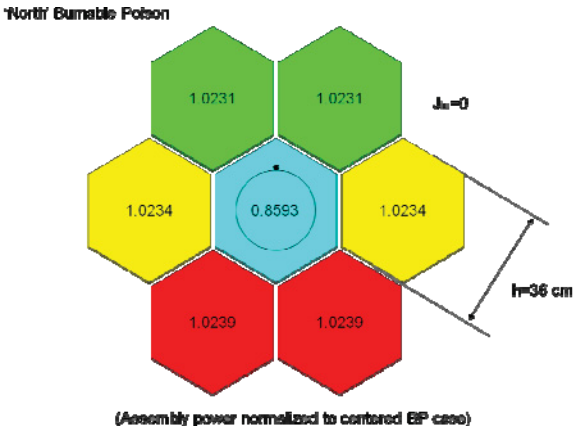

Fig. 5: Simplified mini-core effective multiplication factor and power distribution with 'North' BP.

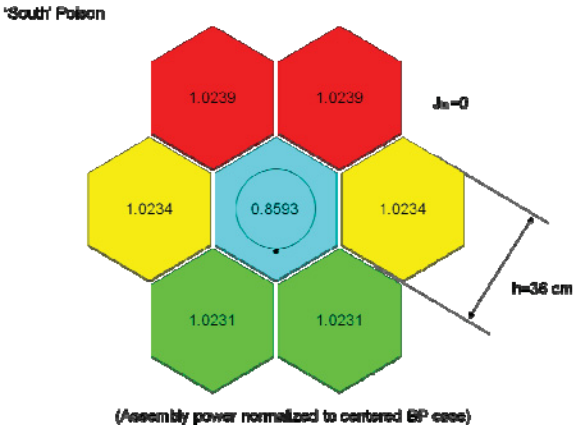

Fig. 6: Simplified mini-core effective multiplication factor and power distribution with 'South' BP.

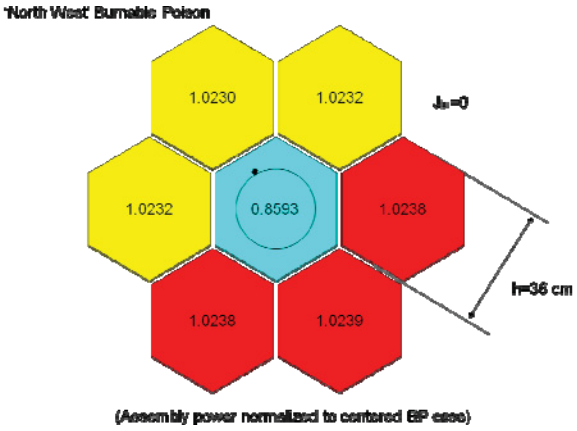

Fig. 7: Simplified mini-core effective multiplication factor and power distribution with 'North West' BP.

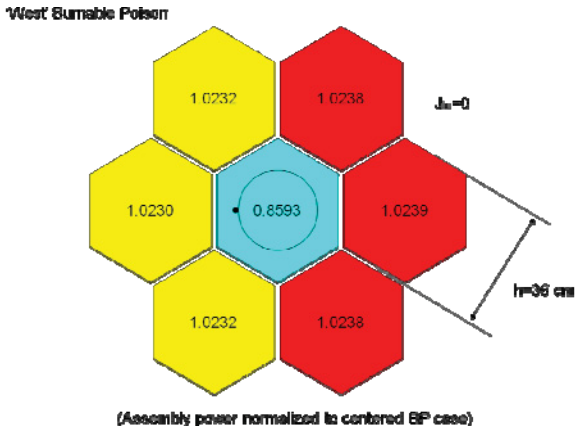

Fig. 8: Simplified mini-core effective multiplication factor and power distribution with 'West' BP.

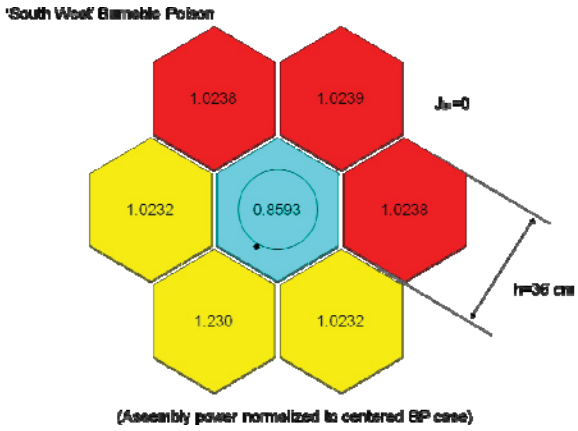

Fig. 9: Simplified mini-core effective multiplication factor and power distribution with 'South West' BP.

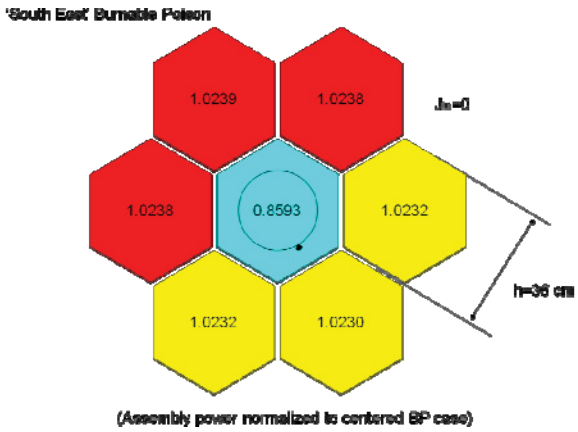

Fig. 10: Simplified mini-core effective multiplication factor and power distribution with 'South East' BP.

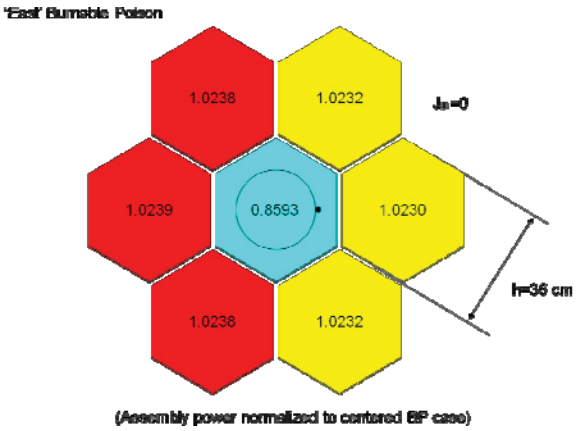

Fig. 11: Simplified mini-core effective multiplication factor and power distribution with 'East' BP.

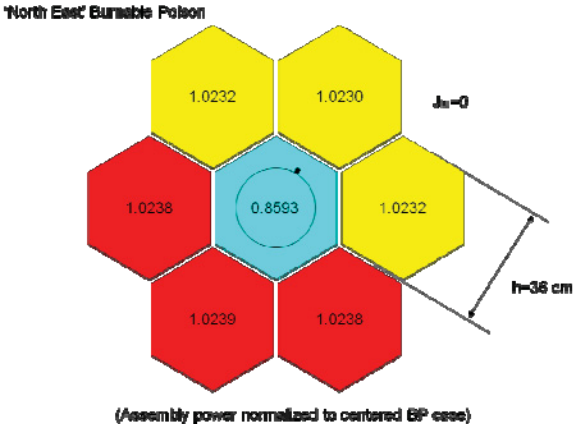

Fig. 12: Simplified mini-core effective multiplication factor and power distribution with 'North East' BP.

VII. CONCLUSION

In this article we have outlined a novel treatment for burnable poisons in hexagonal geometry under the assumption of neutron diffusion theory. Particularly, we adopt the Nodal Green's Function Method approach for solving the diffusion equation since it allows for the solution of more complicated sources beyond low-order polynomials. This is specially warranted in our case, since we model local intra-assembly burnable poisons as delta distributions, thus requiring special treatment. The modeling of burnable poisons as delta distributions is motivated by the physical observation that in neutron-optically thin, graphite-moderated HTRs burnable poisons can affect neighboring assemblies, and thus produce an inter-assembly effect that is uncharacteristic of other reactor types (such as LWRs). The treatment of the burnable poison as a delta distribution is constraint by imposing the

conservation of the physical absorption reaction rate via the neutron balance equation. In order to generate the necessary data, traditional flux-weighted homogenization is used, which produces a separate, absorber-specific cross-section.

The modified neutron diffusion model with a space-dependent removal cross-section was discretized and solved via the NGFM approach. The final set of discrete equations was derived in this article. While the burnable poison adds extra complexity in the derivation of the one-dimensional transverse-averaged solution, it is relatively straightforward to implement and requires only an extra set of data.

A simplified model problem was proposed in order to demonstrate the mathematical model introduced by the introduction of the explicit treatment of the burnable poison as a delta distribution. Thus far, the testing was limited to this 1-group, 2-D problem. Continuing work is planned to focus on testing the approach against more sophisticated, multi-group, spatially heterogeneous problems.

The new method, presented for the first time in this paper, can possibly be generalized to other analytical nodal methods such as the Analytical Nodal Methods (ANM) [14, 15] and the Nodal Integral Method (NIM) [18], even if they are not based on the Green's function approach, provided the methods allow in their respective formulations the explicit incorporation of particular solutions corresponding to a given prescribed source term in the diffusion equation. The method could also be generalized to the Analytical Function Expansion Nodal (AFEN) [19] provided the expansion functions incorporate a term that reflects the same particular solution corresponding to the burnable poison sink in the source term of the diffusion equation. Its generalization to polynomial nodal methods and to finite element methods would be considerably more complicated.

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