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INTRODUCTION

The Method of Manufactured Solutions (MMS) is an accepted technique to verify that a numerical discretization for the radiation transport equation has been implemented correctly [1, 2]. This technique offers a few advantages over other methods such as benchmark problems or analytical solutions. The solution can be manufactured such that properties for the angular flux are either stressed or preserved. For radiation transport, the properties can include desired smoothness, positivity of the angular flux, and an arbitrary order of anisotropy in angle. In addition, the angular flux solution can be manufactured for multidimensional problems where analytical solutions are difficult to obtain in general.

RattleSnake is a S_N radiation transport application built using the Multiphysics Object Oriented Simulation Environment (MOOSE) [3] framework for tightly coupled multiphysics simulations. RattleSnake solves the radiation transport equation in the Self-Adjoint Angular Flux (SAAF) formulation [4] discretized with the multigroup approximation in energy, the continuous FEM (Finite Element Method) in space and S_N in angle. RattleSnake can solve both the source and eigenvalue problems. The built-in transient solvers in MOOSE make solving the transient problem straight-forward. RattleSnake can solve problems containing materials with an arbitrary order of anisotropic scattering cross sections. Utilization of MOOSE simplifies the implementation greatly. Due to the complexity of the discretization schemes and the solving techniques, it is desired to apply MMS as part of the verification procedure of RattleSnake.

In this summary, the MMS technique and the implementation in the SAAF Formulation in RattleSnake is presented. MOOSE includes support functions which help build an MMS source, a supporting interface, and provide the functionality which is well suited to this study.. Results generated using MMS with RattleSnake are shown, which are consistent with the expected error convergence order.

DESCRIPTION OF THE MANUFACTURED SOLUTION FOR THE SELF-ADJOINT FORMULATION IN RATTLESNAKE

In the manufactured solution method, the angular flux is chosen such that desired properties are stressed, and desired boundary conditions are achieved. The solution is generally analytic, and often easy to compute and integrate. For the SAAF method implemented in RattleSnake, the analytical expression for the angular flux is substituted into the first-order form of the transport equation and a radiation source is obtained that gives the desired analytic expression for the angular flux. Thus if the mono-energetic steady state angular flux is manufactured to be:

$$\psi(r,\vec{\Omega}) = \psi_M(r,\vec{\Omega}), \tag{1}$$

for the transport equation:

$$\vec{\Omega}\cdot\nabla\psi(r,\vec{\Omega}) + \Sigma_t\psi(r,\vec{\Omega}) = \frac{1}{4\pi}\int_{\Omega'} d\Omega'\Sigma_s(\vec{\Omega}\cdot\vec{\Omega}')\psi_M(r,\vec{\Omega}') + Q(r,\vec{\Omega}) \tag{2}$$

then the particle source is given by

$$Q_M(r,\vec{\Omega}) = \vec{\Omega}\cdot\nabla\psi_M(r,\vec{\Omega}) + \Sigma_t\psi_M(r,\vec{\Omega}) - \frac{1}{4\pi}\int_{\Omega'} d\Omega'\Sigma_s(\vec{\Omega}\cdot\vec{\Omega}')\psi_M(r,\vec{\Omega}) \tag{3}$$

and the boundary conditions are simply specified by the manufactured solution. If reflective or albedo boundary conditions are desired the manufactured angular flux should be designed such that these boundary conditions are satisfied. Even though the manufactured flux given in Eq. (1) can be relatively simple and smooth the corresponding source given in Eq. (3) can be complicated with different discontinuities caused by changes in the material properties.

The derivation and description of the SAAF formulation and the derivation of the associated weak formulation can be found in the literature [5,6]. The weak form for SAAF in one direction $\vec{\Omega}$ is often simply stated: to find ψ in a function space over the spatial domain *D*, such that

$$b_{\vec{\Omega}}(\psi,\psi^*) = l_{\vec{\Omega}}(\psi^*) \tag{4}$$

for any test functions ψ^* in the same function space, where $b_{\vec{\Omega}}(\psi,\psi^*)$ is the weak bilinear form:

$$b_{\vec{\Omega}}(\psi,\psi^*) = \left(\vec{\Omega}\cdot\vec{\nabla}\psi, \frac{1}{\Sigma_t}\vec{\Omega}\cdot\vec{\nabla}\psi^*\right)_D + \left(\psi,\Sigma_t\psi^*\right)_D + \left\langle\psi,\psi^*\right\rangle^+ \quad (5)$$

and the corresponding right hand side is

$$l_{\vec{\Omega}}(\psi^*) = \left(Q,\psi^* + \frac{1}{\Sigma_t}\vec{\Omega}\cdot\vec{\nabla}\psi^*\right)_D + \left\langle\psi^{inc},\psi^*\right\rangle^- . \quad (6)$$

Notation used in the above two equations are defined as

$$\left(a,b\right)_D \equiv \int_D a(\vec{r})b(\vec{r})d\vec{r}$$

$$\left\langle a,b\right\rangle^{\pm} \equiv \int_{\partial D^{\pm}} \left|\vec{\Omega}\cdot\vec{n}\right| a(\vec{r})b(\vec{r})d\vec{r} ,$$

$\partial D^{\pm}$ are the upwind and downwind boundary with respect to the direction and $\vec{n}$ is the outward unit norm on the boundary. In the SAAF implementation of RattleSnake with MMS, the source in Eq. (6) is the source derived from the manufactured solution given in Eq. (3) plus the scattering contribution, otherwise the source is the direct input.

The manufactured solution given by Eq. (1) and the corresponding source given by Eq. (3) can be designed to include both time, angle and energy dependence.

MOOSE provides a way in which different components of operators can be quickly assembled in a Finite Element Jacobian Free Newton Krylov (JFNK) solver framework. A MMS kernel and interface has been implemented inside of RattleSnake that allows for different manufactured solutions. Manufactured solutions can either be specified as external functions that are compiled with the RattleSnake source or as functions specified in the RattleSnake input file. RattleSnake accepts the manufactured solution as part of the input and outputs the difference between the results and the manufactured solution. Thus RattleSnake can be tested as a whole with arbitrarily manufactured solutions. The construction of the source for the transport equation from the manufactured solution is a component inside RattleSnake which facilitates the test greatly. The focus can then be designing the manufactured functions and verifying the theoretical convergence. This makes testing the code with functions defined in the full phase space easier, and allows for different manufactured solutions to be tested rapidly.

RESULTS

A continuous mono-energetic anisotropic angular flux manufactured solution was implemented inside of RattleSnake for a unit 2D square domain. The manufactured angular flux solution is

$$\psi_M(x,y,\mu,\eta) = (1-\mu^2)(1-\eta^2)\sin(\pi x)\sin(\pi y). \quad (7)$$

The manufactured solution given by Eq. (7) is smooth in space and angle, and gives a scalar flux which is both positive and symmetric over the unit square domain.

To obtain the order of convergence the error is calculated with the L_2 error norm:

$$\left\|\phi-\phi^*\right\|_2 = \sqrt{\int\left(\phi-\phi^*\right)^2} . \quad (8)$$

The scalar flux ϕ is obtained from RattleSnake and the scalar flux ϕ^*(x,y) is the manufactured scalar. A log plot of the error for the order of convergence for linear elements (p=1) is shown in Figure 1.

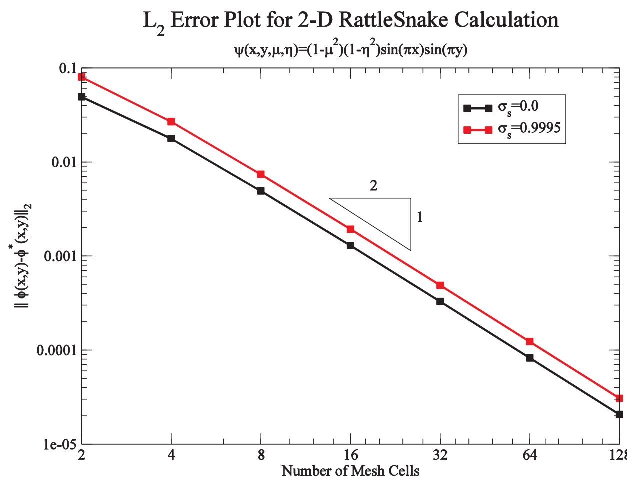

Figure 1. A log plot of the Number of Mesh Cells along an axial direction vs. the L2 Error.

For the convergence results shown in Figure 1, the total cross section was set to 1.0 in each case, and the results were generated on a uniform quadrilateral mesh. The results show the expected 2nd order of convergence. We are working on applying MMS for the multigroup and transient analysis.

CONCLUSIONS

MMS has been incorporated into the core components inside of the RattleSnake radiation transport solver. Since MMS is a component, a manufactured solution can either be included as a compiled external function or specified in the RattleSnake input file. Preliminary results show that RattleSnake obtains the expected order of convergence for an anisotropic angular flux on a uniform quadrilateral mesh.

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